Anticipative optimal investment by rough dynamic programming

Peter Bank

Technische Universität Berlin

based on joint work with
Franziska Bielert

Advances in Financial Mathematics
Jussieu, Paris, January 27–30, 2026

Illustrative problem: Frontrunning with price impact

Bachelier stock price dynamics, zero interest rate:

$$S_t = s_0 + \mu t + \sigma W_t, \quad t \geq 0$$

Information flow: peek Δ time units ahead $\rightsquigarrow$ "frontrunning"

$$\mathscr{G}_t^{\Delta} = \mathscr{F}_{t+\Delta}^{S}, \quad t \geq 0.$$

Egregious arbitrage opportunities

Illustrative problem: Frontrunning with price impact

Bachelier stock price dynamics, zero interest rate:

$$S_t = s_0 + \mu t + \sigma W_t, \quad t \geq 0$$

Information flow: peek Δ time units ahead $\rightsquigarrow$ "frontrunning"

$$\mathscr{G}_t^{\Delta} = \mathscr{F}_{t+\Delta}^{S}, \quad t \geq 0.$$

Egregious arbitrage opportunities curtailed by price impact because execution price is

$$S_t^{\phi} = S_t + \frac{\Lambda}{2}\phi_t$$

where $\phi_t = \frac{d}{dt}\Phi_t$ is the frontrunner's turnover rate $\rightsquigarrow$ temporary price impact à la Almgren-Chriss

Problem formulation: Frontrunning with price impact

Profits and losses from trading according to $\Phi_t = \Phi_0 + \int_0^t \phi_s ds$:

$$V_T^{\Phi_0,\phi} = -\int_0^T S_t^\phi d\Phi_t + \Phi_T S_T - \Phi_0 S_0$$
$$= \Phi_0(S_T - S_0) + \int_0^T \phi_t (S_T - S_t) dt - \frac{\Lambda}{2} \int_0^T \phi_t^2 dt,$$

$\rightsquigarrow$ mark to market stock positions at time 0 and T

Natural class of **admissible strategies**:

$$\mathscr{A}^\Delta = \left\{ \phi = (\phi_t)_{t \in [0,T]} \ \mathscr{G}^\Delta\text{-optional with } \int_0^T \phi_t^2 dt < \infty \ \text{ a.s.} \right\}$$

Exponential **utility maximization**:

$$\text{Maximize } \mathbb{E}\left[u(V_T^{\Phi_0,\phi}) \right] = \mathbb{E}\left[-\exp\left(-\alpha V_T^{\Phi_0,\phi} \right) \right] \text{ over } \phi \in \mathscr{A}^\Delta$$

Optimal Policy

Maximize $\mathbb{E}\left[u(V_T^{\Phi_0,\phi})\right] = \mathbb{E}\left[-\exp\left(-\alpha V_T^{\Phi_0,\phi}\right)\right]$ over $\phi \in \mathscr{A}^\Delta$

Theorem (B., Dolinsky, Rasonyi '22)

The **optimal turnover rate** *at time* $t \in [0, T]$ *is*

$$\hat{\phi}_t = \frac{d}{dt}\hat{\Phi}_t = \frac{1}{\Lambda}\left(\bar{S}_t^\Delta - S_t\right) + \frac{\Upsilon^\Delta(T-t)}{\Delta}\left(\frac{\mu}{\alpha\sigma^2} - \hat{\Phi}_t\right),$$

where $\bar{S}^\Delta$ *denotes the* **stock price average**

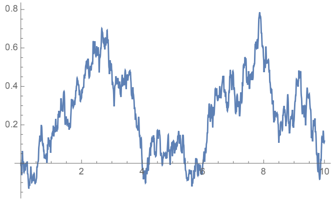
$$\bar{S}_t^\Delta := \left(1 - \Upsilon^\Delta(T-t)\right) S_{(t+\Delta)\wedge T} + \Upsilon^\Delta(T-t)\frac{1}{\Delta}\int_0^\Delta S_{t+s}ds$$

with

$$\Upsilon^\Delta(\tau) := \frac{\Delta\sqrt{\rho}\tanh(\sqrt{\rho}(\tau-\Delta)^+)}{1+\Delta\sqrt{\rho}\tanh(\sqrt{\rho}(\tau-\Delta)^+)}, \quad \tau \geq 0,$$

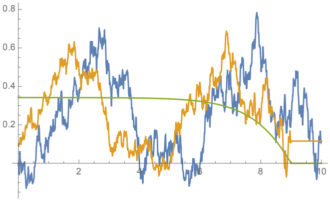
where ρ *is the risk/liquidity ratio* $\rho = \alpha\sigma^2/\Lambda$*.*

Discussion: Optimal policy illustration



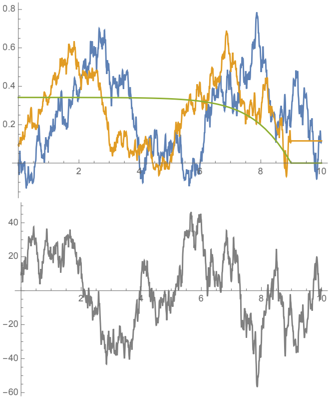
- $S_0 = 0$, $\mu = .1$, $\sigma = .3$, $\alpha = .03$, $T = 10\Delta$, $\Lambda = .01$
- stock price S

Discussion: Optimal policy illustration



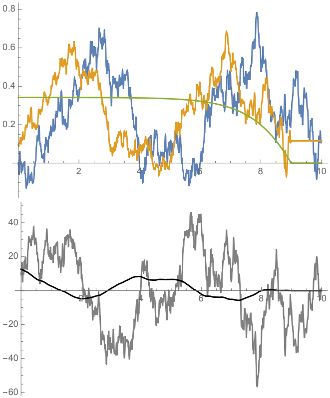
- $S_0 = 0$, $\mu = .1$, $\sigma = .3$, $\alpha = .03$, $T = 10\Delta$, $\Lambda = .01$
- stock price S
- stock price average $\bar{S}^\Delta$
- weight Υ^Δ

Discussion: Optimal policy illustration



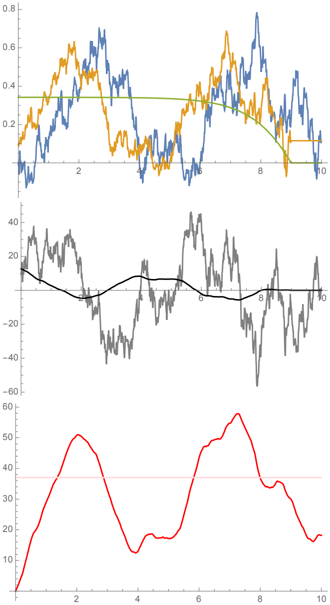
- $S_0 = 0$, $\mu = .1$, $\sigma = .3$, $\alpha = .03$, $T = 10\Delta$, $\Lambda = .01$
- stock price S
- stock price average $\bar{S}^{\Delta}$
- weight Υ^{Δ}
- inside trades $(\bar{S}^{\Delta} - S)/\Lambda$

Discussion: Optimal policy illustration



- $S_0 = 0,\ \mu = .1,\ \sigma = .3,$
  $\alpha = .03,\ T = 10\Delta,\ \Lambda = .01$
- stock price S
- stock price average $\bar{S}^{\Delta}$
- weight Υ^{Δ}
- inside trades $(\bar{S}^{\Delta} - S)/\Lambda$
- trades tracking Merton ratio

Discussion: Optimal policy illustration



- $S_0 = 0$, $\mu = .1$, $\sigma = .3$, $\alpha = .03$, $T = 10\Delta$, $\Lambda = .01$
- stock price S
- stock price average $\bar{S}^{\Delta}$
- weight Υ^{Δ}
- inside trades $(\bar{S}^{\Delta} - S)/\Lambda$
- trades tracking Merton ratio
- optimal total positions $\hat{\Phi}$

Convex duality approach . . .

. . . heavily relying on exponential utility and the Bachelier price dynamics.

Theorem (B., Dolinsky, Rasonyi '22; cf. Guasoni, Rasonyi '15)

We have

$$\max_{\phi \in \mathscr{A}} \left\{ -\frac{1}{\alpha} \log \mathbb{E}\left[\exp\left(-\alpha V_T^{\Phi_0,\phi}\right)\right] \right\}$$
$$= \inf_{\mathbb{Q} \in \mathscr{Q}} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\alpha} \log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) + \Phi_0(S_T - S_0) + \frac{1}{2\Lambda}\int_0^T |\mathbb{E}_{\mathbb{Q}}[S_T|\mathscr{G}_t] - S_t|^2 dt\right].$$

There is a minimizer $\hat{\mathbb{Q}}$ *and it yields via*

$$\hat{\phi}_t = \frac{\mathbb{E}_{\hat{\mathbb{Q}}}[S_T|\mathscr{G}_t] - S_t}{\Lambda}, \quad t \in [0, T],$$

the unique optimal turnover rates for the primal problem. . . .

$\rightsquigarrow$ no (obvious?) way to generalize to more generic settings

Mathematical challenge

How to prove this using a more generic dynamic programming approach?

Main obstacles:

- **path-dependent solution** unavoidable $\rightsquigarrow$ infinite-dimensional HJB approaches (e.g. Fabbri, Gozzi, Swiech '17, ...), path-dependent PDE-literature (e.g. Ekren, Keller, Touzi, Zhang '14, ...)
- **rough semimartingale state dynamics** because of anticipative information $\rightsquigarrow$ rough analysis tools (e.g. Friz, Hairer '20, ...)

Generic anticipative stochastic control problem

Find a control ϕ for the system $Y = Y^{t,y,\phi}$ with dynamics

$$Y(t) = y, \quad dY(s) = b(s, Y(s), \phi(s))ds + f(Y(s))dW(s), \ s \in [t, T]$$ [1]

which yields the maximal expected terminal reward

$$v(t, y) := \operatorname*{ess\,sup}_{\phi \in \mathscr{A}} \mathbb{E}\left[g(Y^{t,y,\phi}(T)) \mid \mathscr{F}^W_{\tau(t)}\right],$$

for some payoff function $g : \mathbb{R} \to [0, \infty)$, where admissible controls controls $\phi \in \mathscr{A}$ are allowed to be **anticipative** in the sense that

$$\phi(t) \in \mathscr{F}^W_{\tau(t)} \text{ for a given increasing } \tau(t) > t, \ t \in [0, T),$$

with $\tau(T) = T$.

Example: $\tau(t) = (t + \Delta) \wedge T$ for some $\Delta > 0$

[1] **For notational simplicity:** All dynamics are written for the one-dimensional case, but everything holds in the multi-dimensional case too

Generic anticipative stochastic control problem

Find a control ϕ for the system $Y = Y^{t,y,\phi}$ with dynamics

$$Y(t) = y, \quad dY(s) = b(s, Y(s), \phi(s))ds + f(Y(s))dW(s), \; s \in [t, T]$$ [1]

which yields the maximal expected terminal reward

$$v(t, y) := \operatorname*{ess\,sup}_{\phi \in \mathscr{A}} \mathbb{E}\left[g(Y^{t,y,\phi}(T)) \mid \mathscr{F}^W_{\tau(t)} \right],$$

for some payoff function $g\colon \mathbb{R} \to [0, \infty)$, where admissible controls controls $\phi \in \mathscr{A}$ are allowed to be **anticipative** in the sense that

$$\phi(t) \in \mathscr{F}^W_{\tau(t)} \text{ for a given increasing } \tau(t) > t, \; t \in [0, T),$$

with $\tau(T) = T$.

Example: $\tau(t) = (t + \Delta) \wedge T$ for some $\Delta > 0$

[1] **For notational simplicity:** All dynamics are written for the one-dimensional case, but everything holds in the multi-dimensional case too

State dynamics as a Rough Differential Equation (RDE)

For any scenario ω, lift $W(\omega)$ to an Ito rough path $\mathbf{W} = (W, \mathbb{W})(\omega)$ (Lyons '98), and, for measurable $\phi : [0, T] \to \mathbb{R}$, consider the deterministic **rough differential equation**

$$\begin{aligned} Y(t) &= y, \\ dY(s) &= b(s, Y(s), \phi(s))ds + f(Y(s))d\mathbf{W}(s), \quad s \in (t, T]. \end{aligned}$$

- **existence and uniqueness** from rough analysis (e.g. Friz, Hairer '20) under regularity assumptions on b and f; control in diffusion part would lead to degeneracies (Allan, Cohen '19)
- **measurable dependence** on problem data (t, y, W, ϕ) from Friz, Le, Zhang '24
- Buckdahn, Ma '07: approach via Doss-Sussmann type flow *transform* of dynamics;
  Friz, Le, Zhang '24: *intrinsic* construction using rough integration

Dynamic programming principle

Under said regularity assumptions on b and f, we have

$$v(t,y) = \operatorname*{ess\,sup}_{\phi \in \mathscr{A}} \mathbb{E}[v(s, Y^{t,y,\phi}(s))|\mathscr{F}_{\tau(t)}] \text{ for } t \leq s$$

also for random starting points $y \in \mathscr{F}_{\tau(t)}$.
$\rightsquigarrow$ **supermartingale dynamics for value processes** induced by admissible controls $\phi \in \mathscr{A}$:

$$v(s, Y^{t,y,\phi}(s)) \geq \mathbb{E}[v(s', Y^{t,y,\phi}(s'))|\mathscr{F}_{\tau(s)}] \text{ for } t \leq s \leq s' \leq T$$

Path-dependency of the value function

$\rightsquigarrow$ We need to understand the (super-) martingale-dynamics of value processes $(v(s, Y^{t,y,\phi}(s)))_{s\in[t,T]}$!

How to represent

$$v(t,y) = \operatorname*{ess\,sup}_{\phi\in\mathscr{A}} \mathbb{E}\left[g(Y^{t,y,\phi}(T))\middle|\mathscr{F}^W_{\tau(t)}\right] \in \sigma((W(s)-W(t))_{s\in(t,\tau(t)]})$$

to make **functional Ito formula** applicable?

Dupire's functional derivatives

Definition (Dupire '09):

$F\colon [0,T] \times D([0,T],\mathbb{R}) \to \mathbb{R}$ called causal if $F(t,X) = F(t, X_{t\wedge\cdot})$

Dupire's functional derivatives

Definition (Dupire '09):

$F\colon [0,T] \times D([0,T],\mathbb{R}) \to \mathbb{R}$ called causal if $F(t,X) = F(t, X_{t\wedge\cdot})$ and causally differentiable if vertical/horizontal perturbations at time t change evaluation F smoothly

$$\partial_X F(t,X) = \frac{\partial}{\partial h}\Big|_{h=0} F(t, X + h1_{[t,T]})$$

$$DF(t,X) = \frac{\partial}{\partial s}\Big|_{s=0+} F(t+s, X_{t\wedge\cdot})$$

Dupire's functional derivatives

Definition (Dupire '09):

$F\colon [0,T]\times D([0,T],\mathbb{R})\to\mathbb{R}$ called causal if $F(t,X)=F(t,X_{t\wedge\cdot})$ and causally differentiable if vertical/horizontal perturbations at time t change evaluation F smoothly

$\partial_X F(t,X)=\frac{\partial}{\partial h}|_{h=0}F(t,X+h1_{[t,T]})$

$DF(t,X)=\frac{\partial}{\partial s}|_{s=0+}F(t+s,X_{t\wedge\cdot})$

$\rightsquigarrow$ $v(t,y)\in\sigma((W(s)-W(t))_{s\in(t,\tau(t)]})$ expected to depend ...

causally on $W_t^{\circ\tau}:=(W(\tau(s))-W(\tau(0)))_{s\in[0,t]}$

parameter wise on initial segment $W^{\tau(0)}:=W_{|[0,\tau(0)]}$

causally on $W(t)$

Rough functional Ito formula

Theorem (Bielert '26, see also Cuchiero, Guo, Primavera '25)

For a sufficiently regular causal functional F *and an* α*-Hölder-continuous path* X*, we have for* $n > (1/\alpha - 1)^2$ *that*

$$\int_0^T \partial_X F(s, X) d\mathbf{X}(s) := \lim_{|\mathscr{P}| \to 0} \sum_{[s,t] \in \mathscr{P}} \sum_{k=1}^{n} \partial_X^k F(s, X)(X(t) - X(s))^k / k!$$

exists and gives

$$F(T, X) = F(0, X) + \int_0^T DF(s, X) ds + \int_0^T \partial_X F(s, X) d\mathbf{X}(s).$$

Rough functional Ito formula for X with Brownian driver

Theorem

For $(1/2-)$-Hölder continuous paths X with quadratic variation $[X]$ and $F \in \mathbb{C}_b^{1,4}$,

$$\int_0^T \partial_X F(s,X) d\mathbf{X}(s) := \lim_{|\mathscr{P}|\to 0} \sum_{[s,t]\in\mathscr{P}} \Big\{ \partial_X F(s,X)(X(t)-X(s)) + \frac{1}{2}\partial_X^2 F(s,X)\left((X(t)-X(s))^2 - ([X](t)-[X](s))\right)\Big\}$$

exists and gives

$$F(T,X) = F(0,X) + \int_0^T DF(s,X)ds + \int_0^T \frac{1}{2}\partial_X^2 F(s,X)d[X](s) + \int_0^T \partial_X F(s,X) d\mathbf{X}(s).$$

Hamilton-Jacobi-Bellman equation

Expect

$$v(t, y) = u(t, y, W(t), W^{\tau(0)}, W_t^{\circ\tau})$$

for some **deterministic** mapping

$$\begin{aligned} u \colon [0, T] \times \mathbb{R} \times \mathbb{R} \times C([0, \tau(0)], \mathbb{R}) \times D([0, T], \mathbb{R}) &\to \mathbb{R} \\ (t, y, w, \mathrm{w}, \mathrm{z}) &\mapsto u(t, y, w, \mathrm{w}, \mathrm{z}). \end{aligned}$$

Hamilton-Jacobi-Bellman equation

Expect

$$v(t,y) = u(t,y,W(t),W^{\tau(0)},W_t^{\circ\tau})$$

for some **deterministic** mapping

$$\begin{aligned} u\colon [0,T]\times\mathbb{R}\times\mathbb{R}\times C([0,\tau(0)],\mathbb{R})\times D([0,T],\mathbb{R}) &\to \mathbb{R} \\ (t,y,w,\mathrm{w},\mathrm{z}) &\mapsto u(t,y,w,\mathrm{w},\mathrm{z}). \end{aligned}$$

such that for a.e. Brownian $\mathrm{w}\in C([0,T],\mathbb{R})$, all $t\in[0,T)$, $y\in\mathbb{R}$,

$$\begin{aligned} 0 &= Du(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + \sup_{\varphi\in\mathbb{R}}\{b(t,y,\varphi)\partial_y u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})\} \\ &\quad + \frac{1}{2}f(y)^2\partial_y^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + f(y)\partial_{yw}u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})) \\ &\quad + \frac{1}{2}\partial_w^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})) + \frac{1}{2}\partial_{\mathrm{z}}^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}))\tau'(t), \\ 0 &= f(y)\partial_y u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + \partial_w u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}), \\ g(y) &= u(T,y,\mathrm{w}(T),\mathrm{w}^{\tau(0)},\mathrm{w}_T^{\circ\tau}). \end{aligned}$$

Hamilton-Jacobi-Bellman equation

Expect

$$v(t,y) = u(t,y,W(t),W^{\tau(0)},W_t^{\circ\tau})$$

for some **deterministic** mapping

$$\begin{aligned} u\colon [0,T]\times\mathbb{R}\times\mathbb{R}\times C([0,\tau(0)],\mathbb{R})\times D([0,T],\mathbb{R}) &\to \mathbb{R} \\ (t,y,w,\mathrm{w},\mathrm{z}) &\mapsto u(t,y,w,\mathrm{w},\mathrm{z}). \end{aligned}$$

such that for a.e. Brownian $\mathrm{w}\in C([0,T],\mathbb{R})$, all $t\in[0,T)$, $y\in\mathbb{R}$,

$$\begin{aligned} 0 &= Du(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + \sup_{\varphi\in\mathbb{R}}\{b(t,y,\varphi)\partial_y u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})\} \\ &\quad + \frac{1}{2}f(y)^2\partial_y^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + f(y)\partial_{yw}u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})) \\ &\quad + \frac{1}{2}\partial_w^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})) + \frac{1}{2}\partial_{\mathrm{z}}^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}))\tau'(t), \\ 0 &= f(y)\partial_y u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + \partial_w u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}), \\ g(y) &= u(T,y,\mathrm{w}(T),\mathrm{w}^{\tau(0)},\mathrm{w}_T^{\circ\tau}). \end{aligned}$$

Hamilton-Jacobi-Bellman equation

Expect

$$v(t,y) = u(t,y,W(t),W^{\tau(0)},W_t^{\circ\tau})$$

for some **deterministic** mapping

$$u\colon [0,T]\times\mathbb{R}\times\mathbb{R}\times C([0,\tau(0)],\mathbb{R})\times D([0,T],\mathbb{R})\to\mathbb{R}$$
$$(t,y,w,\mathrm{w},\mathrm{z})\mapsto u(t,y,w,\mathrm{w},\mathrm{z}).$$

such that for a.e. Brownian $\mathrm{w}\in C([0,T],\mathbb{R})$, all $t\in[0,T)$, $y\in\mathbb{R}$,

$$\begin{aligned}
0 &= Du(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + \sup_{\varphi\in\mathbb{R}}\{b(t,y,\varphi)\partial_y u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})\}\\
&\quad + \frac{1}{2}f(y)^2\partial_y^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + f(y)\partial_{yw}u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}))\\
&\quad + \frac{1}{2}\partial_w^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})) + \frac{1}{2}\partial_{\mathrm{z}}^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}))\tau'(t),\\
0 &= f(y)\partial_y u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + \partial_w u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}),\\
g(y) &= u(T,y,\mathrm{w}(T),\mathrm{w}^{\tau(0)},\mathrm{w}_T^{\circ\tau}).
\end{aligned}$$

Comparison in case of full information

Full information case $\tau(t) \equiv T$ corresponds to **stochastic pathwise optimal control problem**:

- Lions, Souganides '98: purely analytic viscosity solution approach
- Buckdahn, Ma '07: Doss-Sussmann transform
- Friz, Le, Zhang '24: intrinsic notion

Give meaning to what is now called a **rough HJB equation**:

$$
\begin{aligned}
0 = d_t v(t,y) + \Big(\sup_{\varphi} \{ b(t,y,\varphi) \partial_y v(t,y) \} + \frac{1}{2} f(y)^2 \partial_y^2 v(t,y) \Big) dt \\
+ f(y) \partial_y v(t,y) dW_t
\end{aligned}
$$

$$
g(y) = v(T,y)
$$

A verification theorem

Theorem

Under suitable regularity assumptions, if u *satisfies the HJB equation then almost surely*

$$v(t, y) \le u(t, y, W(t), W^{\tau(0)}, W_t^{\circ\tau})$$

for all $t \in [0, T)$, $y \in \mathbb{R}$, *and equality holds true at* (t, y) *if, in addition, there is an admissible control* $\phi^* \in \mathscr{A}$ *such that* $ds \otimes \mathbb{P}(d\omega)$*-almost everywhere* $\varphi^* := \phi^*(s, \omega)$ *attains*

$$\sup_{\varphi \in \mathbb{R}} \left\{ \partial_y u(s, Y^{t,y,\phi^*}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) b(s, Y^{t,y,\phi^*}(s), \varphi) \right\}.$$

This control ϕ^* *is then optimal when starting in* y *at time* t.

A (partially) converse verification theorem

Theorem

If the value function is of the form $v(t,y) = u(t,y,W(t),W^{\tau(0)},W_t^{\circ\tau})$ *for some sufficiently regular* u*, then this* u *must solve the causal HJB equation, i.e., for a.e. Brownian* $\mathrm{w} \in C([0,T],\mathbb{R})$*, all* $t \in [0,T)$*,* $y \in \mathbb{R}$*,*

$$\begin{aligned}
0 &= Du(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) \\
&\quad + \sup_{\varphi\in\mathbb{R}}\{\partial_y u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})b(t,y,\varphi)\} \\
&\quad + \frac{1}{2}f(y)^2\partial_y^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}) + f(y)\partial_{yw}u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})) \\
&\quad + \frac{1}{2}\partial_w^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})) + \frac{1}{2}\partial_z^2 u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}))\tau'(t), \\
0 &= f(y)\partial_y u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau})\rangle + \partial_w u(t,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}), \\
g(y) &= u(T,y,\mathrm{w}(t),\mathrm{w}^{\tau(0)},\mathrm{w}^{\circ\tau}).
\end{aligned}$$

About that transport equation . . .

Heuristic dynamics of the value process:

$$
\begin{aligned}
& dv(s, Y^{t,y,\phi}(s)) \\
& = \Big\{ Du(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W^{\circ\tau}_s) + \partial_y u \dots \Big\} ds \\
& \quad + \Big\{ f(y)\partial_y u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W^{\circ\tau}_s) \\
& \qquad\qquad + \partial_w u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W^{\circ\tau}_s) \Big\} d\mathbf{W}(s) \\
& \quad + \Big\{ \partial_z u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W^{\circ\tau}_s) \Big\} dW^{\circ\tau}(s)
\end{aligned}
$$

About that transport equation . . .

Heuristic dynamics of the value process:

$$
\begin{aligned}
& dv(s, Y^{t,y,\phi}(s)) \\
& = \Big\{ Du(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) + \partial_y u \dots \Big\} ds \\
& \quad + \Big\{ f(y)\partial_y u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \\
& \qquad\qquad + \partial_w u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \Big\} d\mathbf{W}(s) \\
& \quad + \Big\{ \partial_z u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \Big\} dW^{\circ\tau}(s)
\end{aligned}
$$

- $\{\dots\} dW^{\circ\tau}(s)$ gives martingale dynamics because $W^{\circ\tau} = W(\tau(.))$ is an $(\mathscr{F}_{\tau(s)})_{s\in[t,T]}$-martingale

About that transport equation ...

Heuristic dynamics of the value process:

$$
\begin{aligned}
& dv(s, Y^{t,y,\phi}(s)) \\
&= \Big\{ Du(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) + \partial_y u \ldots \Big\} ds \\
&\quad + \Big\{ f(y) \partial_y u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \\
&\qquad\qquad + \partial_w u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \Big\} d\mathbf{W}(s) \\
&\quad + \Big\{ \partial_z u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \Big\} dW^{\circ\tau}(s)
\end{aligned}
$$

- $\{\ldots\} dW^{\circ\tau}(s)$ gives martingale dynamics because $W^{\circ\tau} = W(\tau(.))$ is an $(\mathscr{F}_{\tau(s)})_{s\in[t,T]}$-martingale
- $\{\ldots\} d\mathbf{W}(s)$-term would yield a rough drift: $\rightsquigarrow$ incompatible with super-martingale dynamics of value process from the dynamic programming principle

About that transport equation ...

Heuristic dynamics of the value process:

$$
\begin{aligned}
& dv(s, Y^{t,y,\phi}(s)) \\
&= \Big\{ Du(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) + \partial_y u \dots \Big\} ds \\
&\quad + \Big\{ f(y)\partial_y u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \\
&\qquad\qquad + \partial_w u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \Big\} d\mathbf{W}(s) \\
&\quad + \Big\{ \partial_z u(s, Y^{t,y,\phi}(s), W(s), W^{\tau(0)}, W_s^{\circ\tau}) \Big\} dW^{\circ\tau}(s)
\end{aligned}
$$

- $\{\dots\}dW^{\circ\tau}(s)$ gives martingale dynamics because $W^{\circ\tau} = W(\tau(.))$ is an $(\mathscr{F}_{\tau(s)})_{s\in[t,T]}$-martingale
- $\{\dots\}d\mathbf{W}(s)$-term would yield a rough drift: $\rightsquigarrow$ incompatible with super-martingale dynamics of value process from the dynamic programming principle
- $\{\dots\}ds \leq 0$ gives classical part of HJB equation

A peek under the hood ...

To make the heuristic interpretation of value process dynamics rigorous make precise, e.g., that, for $s \in [t, \tau(t)]$,

$$\left[\mathbb{E}\Big[\int_t^{\cdot} \partial_w u(r, Y_r^{t,y,\phi}, W(r), W^{\tau(0)}, W_r^{\circ\tau}) d\mathbf{W}(r) \Big| \mathscr{F}_{\tau(t)}\Big]\right](s)$$
$$= \int_t^s \mathbb{E}\left[\partial_w u(r, Y_r^{t,y,\phi}, W(r), W^{\tau(0)}, W_r^{\circ\tau}) \Big| \mathscr{F}_{\tau(t)}\right]^2 dr$$

without running into issues with null sets ...

Illustration in anticipative optimal investment problem

The value $v = v(t, x, \Phi)$ should be given by a causal functional $u = u(t, x, \Phi, s, w, z)$ satisfying the causal HJB equation

$$\begin{cases} 0 & = Du + \sup_{\varphi \in \mathbb{R}} \{ -\frac{\Lambda}{2} \varphi^2 \partial_x u + \varphi \partial_\Phi u \} \\ & \qquad + \frac{1}{2}(\Phi^2 \partial_x^2 u + 2\Phi \partial_{xs}^2 u + \partial_s^2 u + \partial_z^2 u), \\ 0 & = \Phi \partial_x u + \partial_s u, \\ g(x) & = u(T, x, \cdot). \end{cases}$$

The paper in the arXiv gives an explicit solution . . .

Explicit solution of causal HJB equation

For $\mu = 0$, $\sigma = 1$, $\alpha = 1$, the causal HJB equation in the illustrative example is solved by u such that almost surely

$$
\begin{aligned}
&u(t, x, \Phi, W(t), W^{\tau(0)}, W^{\circ\tau}) \\
&= -\exp\Bigg(-x - \frac{\Delta}{2\Lambda}\Omega(t) - \Phi(W((t+\Delta)\wedge T) - W(t)) \\
&\qquad - \frac{1}{2\Lambda}\int_t^{(t+\Delta)\wedge T} (W((t+\Delta)\wedge T) - W(s))^2 ds \\
&\qquad + \frac{1}{2}\Upsilon(t)\Big(\Phi + \frac{1}{\Lambda}\int_t^{(t+\Delta)\wedge T} (W((t+\Delta)\wedge T) - W(s)) ds\Big)^2 \Bigg)
\end{aligned}
$$

and the optimal turnover rate is given by

$$
\begin{aligned}
\hat{\phi}(t) = \frac{1}{\Lambda}\Bigg(&W((t+\Delta)\wedge T) - W(t) \\
&- \Upsilon(t)\Big(\hat{\Phi}(t) + \frac{1}{\Lambda}\int_t^{(t+\Delta)\wedge T} W((t+\Delta)\wedge T) - W(u) du\Big)\Bigg).
\end{aligned}
$$

The latter recovers the result from B., Dolinsky, Rasonyi '22.

Conclusion and outlook

- anticipative optimal investment problem explicitly solved by rough dynamic programming
- state dynamics controlled by tools from rough analysis
- causal HJB equation from functional rough Ito formula including transport equation
- present work: problems with rough stochastic differential equation dynamics (inspired by B., Dolinsky AMO'24)
- use functional Ito-formula of Cuchiero, Guo, Primavera '25 to treat problems with more intricate path-dependencies

Conclusion and outlook

- anticipative optimal investment problem explicitly solved by rough dynamic programming
- state dynamics controlled by tools from rough analysis
- causal HJB equation from functional rough Ito formula including transport equation
- present work: problems with rough stochastic differential equation dynamics (inspired by B., Dolinsky AMO'24)
- use functional Ito-formula of Cuchiero, Guo, Primavera '25 to treat problems with more intricate path-dependencies

Thank you very much!