Robust Pricing & Hedging Duality for American Options in Continuous Time

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joint work with Ivan Guo, arXiv:2510.05463

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Robust pricing and hedging: American options

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Super-hedging price:

$$\pi^A(Z) := \inf\{x : \exists (p, q) \text{ s.t.}$$
$$x + \int_0^u p \cdot dX_t + \int_u^1 q^u \cdot dX_t \ge Z_u, \forall u \in [0,1], \mathcal{Q}\text{-q.s.}\}.$$

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We want duality $\pi^A(Z) = \sup_{\tau\in\mathcal{T},\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau,\cdot)$.

Discrete time picture is subtle but well understood: *duality holds and/or can be recovered*.

Continuous time: *largely open until now*.

Semimartingale Optimal transport

Optimal transport — discrete and continuous

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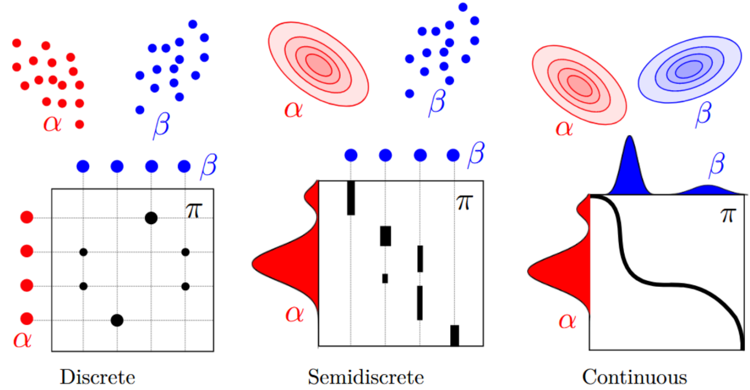

Image: "Computational optimal transport", Gabriel Peyré and Marco Cuturi

Optimal transport — interpolating distributions

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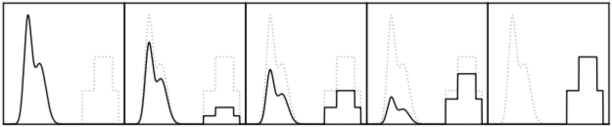
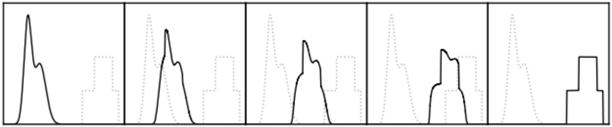


(a) A linear interpolation or "fade"



(b) Displacement interpolation via optimal transport or a "portamento"

Figure 1: The distribution on the left is transformed into the distribution on the right with two different interpolation methods.

Image: "Audio transport: a generalized portamento via optimal transport", Trevor Henderson and Justin Solomon (2019)

Optimal transport — interpolating distributions

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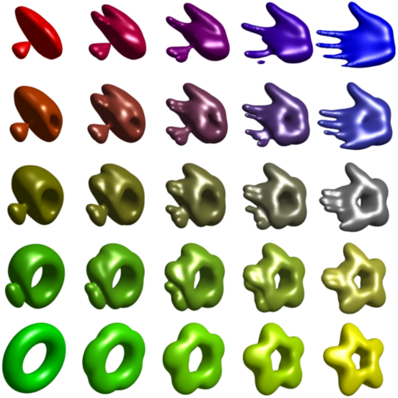

Image: "A short story on optimal transport and its many applications", Filippo Santambrogio

Optimal transport — colour transfer

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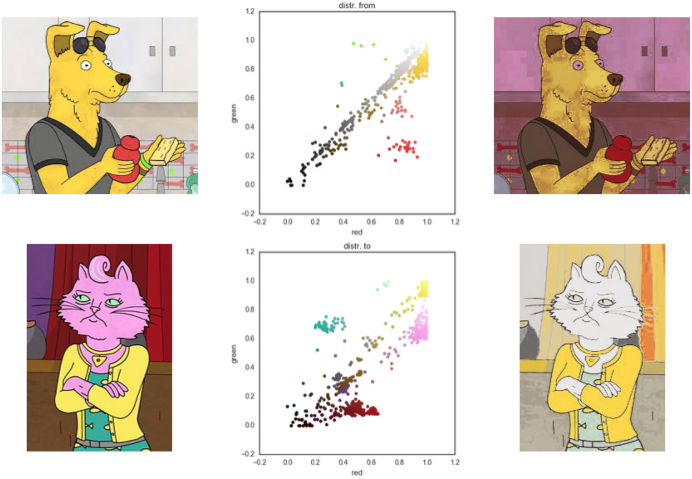

Image: “Notes on optimal transport”, Michiel Stock

Kantorovich duality

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Let $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$ be probability distributions and c be a cost function. The Monge-Kantorovich problem is

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times Y} c(x,y)\, d\pi(x,y)$$

where $\Pi(\mu,\nu)$ is the set of couplings with μ and ν as marginals.

Theorem (Kantorovich duality)

Let $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$, where X, Y are Polish. Suppose the cost function $c : X \times Y \to \mathbb{R}_+ \cup \{\infty\}$ is lower semi-continuous. Then

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times Y} c(x,y)\, d\pi(x,y) = \sup_{\varphi(x)+\psi(y) \le c(x,y)} \int_X \varphi(x)\, d\mu(x) + \int_Y \psi(y)\, d\nu(y)$$

where $\Pi(\mu,\nu)$ is the set of couplings and φ, ψ are measurable functions. Moreover, the infimum is attained.

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The key idea is to characterise the set $\Pi(\mu, \nu)$ via Lagrange multipliers

$$\sup_{\varphi,\psi\in\mathcal{C}} \langle \mu - \pi_x, \varphi\rangle + \langle \nu - \pi_y, \psi\rangle = \begin{cases} 0, & \pi_x = \mu, \pi_y = \nu, \\ \infty, & \text{otherwise}, \end{cases}$$

where π_x, π_y are marginals of π.

A sketch of the proof of the duality:

$$\begin{aligned} \inf_{\pi\in\Pi(\mu,\nu)} \langle \pi, c\rangle &= \inf_{\pi\in\mathcal{M}_+} \sup_{\varphi,\psi\in\mathcal{C}} \langle \pi, c\rangle + \langle \mu - \pi_x, \varphi\rangle + \langle \nu - \pi_y, \psi\rangle \\ &= \sup_{\varphi,\psi\in\mathcal{C}} \inf_{\pi\in\mathcal{M}_+} \langle \pi, c\rangle + \langle \mu - \pi_x, \varphi\rangle + \langle \nu - \pi_y, \psi\rangle \\ &= \sup_{\varphi,\psi\in\mathcal{C}} \inf_{\pi\in\mathcal{M}_+} \langle \pi, c - \varphi(x) - \psi(y)\rangle + \langle \mu, \varphi\rangle + \langle \nu, \psi\rangle \\ &= \sup_{\varphi(x)+\psi(y)\le c} \langle \mu, \varphi\rangle + \langle \nu, \psi\rangle \end{aligned}$$

(Tan & Touzi (2013); Huesmann & Trevisan (2017); Backhoff-Veraguas, Beiglböck, Huesmann & Källblad (2017), Guo & Loeper (2021), ...)

Consider probability measures $\mathbb{P}$ such that X is a semimartingale,

$$X_t = X_0 + \int_0^t \alpha_s^{\mathbb{P}}\, ds + M_t, \quad \langle X\rangle_t = \langle M\rangle_t = \int_0^t \beta_s^{\mathbb{P}}\, ds, \quad \mathbb{P}\text{-a.s.},$$

We say $\mathbb{P}$ has characterstics $(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$.

Semimartingale optimal transport problem

We want to minimise

$$V(\mu_0, \mu_1) = \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} \mathbb{E}^{\mathbb{P}} \int_0^1 H(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dt,$$

where $\mathcal{P}(\mu_0, \mu_1)$ contains probability measures satisfying

$$\mathbb{P} \circ X_0^{-1} = \mu_0, \quad \mathbb{P} \circ X_1^{-1} = \mu_1.$$

The cost function H is convex in (α, β) and may depend on (t, X).

Instead of marginal constraints μ_0 and μ_1, how about other types of constraints? For example:

$$\mathbb{E}^{\mathbb{P}} X_1 = c, \quad \mathbb{E}^{\mathbb{P}} G(X) = c, \quad \mathbb{P} \circ G^{-1} = \rho, \quad \mathbb{P}(G(X) \leq 0) \leq c, \quad \text{etc.}$$

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General abstract constraints

Let $\mathcal{N} \subseteq \mathcal{P}$ be a closed convex set of probability measures. We define

$$F^*(\mu) = \sup_{\psi \in C_b(\Omega)} \int_\Omega \psi \, d\mu - F(\psi) = \begin{cases} 0, & \mu \in \mathcal{N}, \\ +\infty, & \mu \notin \mathcal{N}. \end{cases}$$

This function penalises measures outside $\mathcal{N}$. Some examples:

$$\mathbb{E}^{\mathbb{P}} G(X) = c \quad \Longrightarrow \quad F^*(\mu) = \sup_{\lambda \in \mathbb{R}^m} \lambda \cdot (c - \mathbb{E}^{\mu}(G(X))),$$

$$\mathbb{P} \circ G^{-1} = \rho \quad \Longrightarrow \quad F^*(\mu) = \sup_{\lambda \in C_b(\mathbb{R}^m)} \int_{\mathbb{R}^m} \lambda (d\rho - d\mu).$$

Path-derivatives and functional Itô formula

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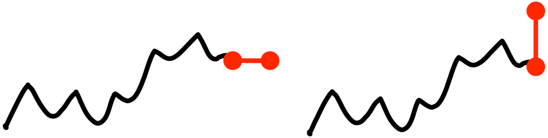
Dupire (2009), Cont & Fournié (2013) introduced non-anticipative path-derivatives operating on $C^{1,2}(\Lambda)$, which are functions of adapted paths. Time derivative $\mathcal{D}_t$ and space derivatives ∇_x, ∇_x^2:



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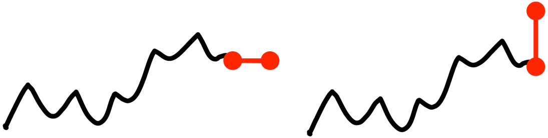

The Functional Itô formula holds for any semimartingale measure $\mathbb{P}$.

$$\phi(t,X) - \phi(0,X) = \int_0^t \mathcal{D}_t\phi\, dt + \nabla_x\phi \cdot dX_t + \frac{1}{2}\nabla_x^2\phi : d\langle X\rangle_t, \quad \mathbb{P}\text{-a.s.}$$

It also serves as a Lagrange multiplier for semimartingale measures.

Main duality result (Guo & Loeper (2021))

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$$V = \inf_{\mathbb{P}\in\mathcal{P}(\rho_0)} F^*(\mathbb{P}) + \int_\Lambda H(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dt d\mathbb{P} \quad \text{s.t. } X \text{ is a } \mathbb{P}\text{-semimartingale}$$

$$= \inf_{\mu,\nu,\alpha,\beta} \sup_{\phi,\psi} \Big(\langle \psi, \mu\rangle - F(\psi) \Big) + \langle H(\alpha,\beta), \nu\rangle$$

$$+ \Big(\langle \phi_1, \mu\rangle - \langle \phi_0, \rho_0\rangle - \langle \mathcal{D}_t\phi + \alpha\cdot\nabla_x\phi + \frac{1}{2}\beta : \nabla_x^2\phi, \nu\rangle \Big)$$

$$= \sup_{\phi,\psi} \inf_{\mu,\nu,\alpha,\beta} \langle \psi + \phi_1, \mu\rangle - F(\psi) - \langle \phi_0, \rho_0\rangle$$

$$- \langle \mathcal{D}_t\phi + \alpha\cdot\nabla_x\phi + \frac{1}{2}\beta : \nabla_x^2\phi - H(\alpha,\beta), \nu\rangle$$

$$= \sup_{\psi\in C_b(\Omega), \phi\in C^{1,2}(\Lambda)} -F(\psi) - \int_{\Omega_0} \phi(0,\cdot)\, d\rho_0,$$

$$\text{s.t.} \quad \phi(1,\cdot) \geq -\psi \quad \text{and} \quad \mathcal{D}_t\phi + H^*\Big(\nabla_x\phi, \frac{1}{2}\nabla_x^2\phi\Big) \leq 0.$$

Dualities in financial mathematics

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$$\inf_{\mathbb{P}\in\mathcal{P}} F^*(\mathbb{P}) + \mathbb{E}^{\mathbb{P}} \int H(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dt = \sup_{\phi \in C^{1,2}(\Lambda)} -F(-\phi(1,\cdot)) - \phi(0,\cdot),$$

$$\text{s.t.} \quad \mathcal{D}_t\phi + H^*\left(\nabla_x\phi, \frac{1}{2}\nabla_x^2\phi\right) = 0.$$

Topological dualities between

- Functions (controls and reward structures, e.g., options, trading portfolios)
- Measures (probabilistic uncertainties, e.g., models, wealth distributions).

Many applications:

- Model calibration — models vs. option prices
- Generative modelling (recall Nizar's talk)
- **Robust finance** — model uncertainty vs. hedging portfolios
- Portfolio selection — wealth distributions vs. utility functions

Robust pricing and hedging of American options in continuous time

Robust European options: model uncertainty

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Consider a market with stocks X. Let $\mathcal{Q} \subset \mathcal{P}$ be the set of possible "models": X is martingale & volatility satisfies (given, pathwise) constraints.

Robust European options: model uncertainty

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Consider a European claim Z. Worst case model price:

$$\sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z.$$

Super-hedging price:

$$\pi(Z) := \inf \left\{ x : \exists q, \text{ s.t. } x + \int_0^1 q \cdot dX_t \geq Z, \ \mathcal{Q}\text{-q.s.} \right\}.$$

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Pricing-hedging dualities

$$\sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z = \pi(Z)$$

are obtained in various settings by Denis & Martini (2006); Soner, Touzi & Zhang (2013); Neufeld & Nutz (2013); Possamaï, Royer & Touzi (2013); Dolinsky & Soner (2014), Perkowski & Prömel (2016), Hou & O. (2018)...

Restricting the drift α to 0 and the diffusion β to a compact set (volatility constraint), the path-dependent OT duality gives

$$\sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z = \inf_{\phi\in C^{1,2}(\Lambda)} \phi(0, X_0),$$

$$\text{s.t.} \quad \phi(1,\cdot) \geq Z \quad \text{and} \quad \mathcal{D}_t\phi + \sup_{\beta\in D} \frac{1}{2}\nabla_x^2\phi : \beta \leq 0. \qquad (1)$$

Each ϕ is actually a super-hedge. For every $\mathbb{P}\in\mathcal{Q}$

$$\begin{aligned} Z &\leq \phi(1, X) \\ &= \phi(0, X_0) + \int_0^1 (\mathcal{D}_t\phi + \frac{1}{2}\beta^{\mathbb{P}} : \nabla_x^2\phi)dt + \nabla_x\phi\cdot dX_t, \quad \mathbb{P}\text{-a.s.} \\ &\leq \phi(0, X_0) + \int_0^1 \nabla_x\phi\cdot dX_t. \end{aligned}$$

Hence $\phi(0, X_0) \geq \pi(Z)$. Since this works for all ϕ satisfying (1), it implies

$$\sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z = \inf_{\phi\in C_0^{1,2}(\Lambda),(1)} \phi(0, X_0) \geq \pi(Z) \geq \sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z.$$

Robust American options

Let Z be an American-style claim. Worst case model price:

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Again, we want duality $\pi^A(Z) = \sup_{\tau\in\mathcal{T},\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau,\cdot)$.

In discrete time, dualities are obtained by Dolinsky (2014); Hobson & Neuberger (2017); Bayraktar & Zhou (2017); Aksamit, Deng, O. & Tan (2019); and more.

Some relevant works in continuous time include Herrmann & Stebegg (2017); Tiplea (2019); Grigorova, Quenez & Sulem (2021) etc.

Enlarged space

The key is to enlarge the space Ω with the stopping decisions to obtain $\bar{\Omega}$, see El Karoui & Tan (2013). Let $\bar{\Omega} := \Theta \times \Omega$ where

$$\Theta := \{\vartheta \in C([0,1],\mathbb{R}) : \vartheta_t = \theta \wedge t, \text{ for some } \theta \in [0,1]\}.$$

Θ is isometric to $[0,1]$, representing the stopping decision.

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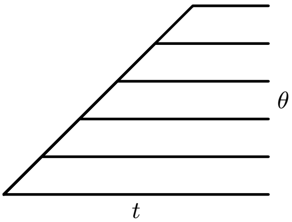

Many aspects of Ω naturally extend to $\bar{\Omega}$, e.g., semimartingale properties (ϑ has characteristics $(\mathbb{1}_{t \le \theta}, 0)$), and the semimartingale optimal transport duality.

Visualising the enlarged space

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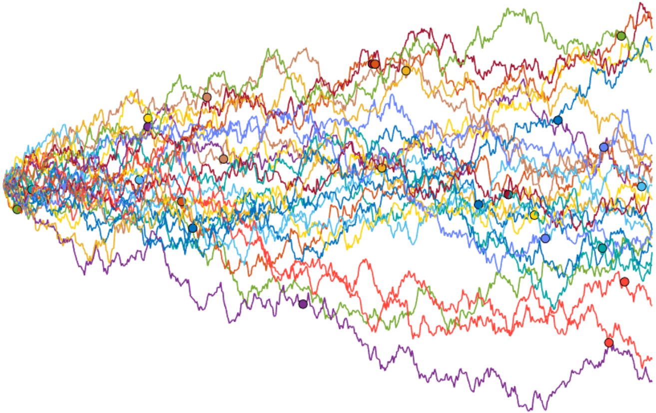

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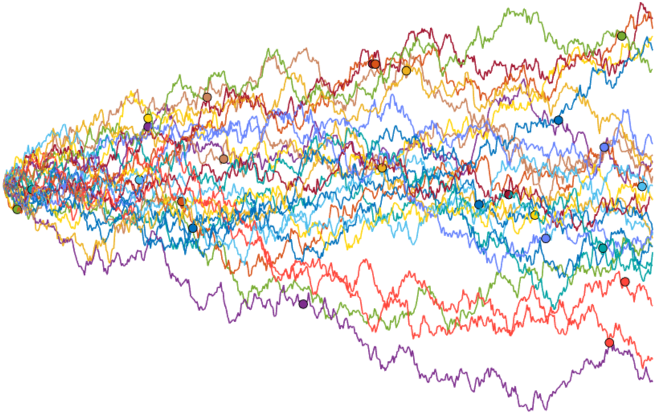

Discrete vs continuous time

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In a discrete time setting, Aksamit et al. (2019) showed using dynamic programming and backward induction that American options are equivalent to European options on the enlarged space $\bar{\Omega}$ and deduced pricing-hedging duality.

$$\bar{\pi}(Z) \geq \pi^A(Z) \geq \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot), \qquad \text{(“easy”)}$$

$$\bar{\pi}(Z) = \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z, \qquad \text{(European duality)}$$

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot). \qquad \text{(“difficult”)}$$

In continuous time settings, we cannot use backward induction, so the last step is problematic.

Convexifying measures and stopping times

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We want to prove $\sup_{\bar{\mathbb{P}}\in\bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z(\theta,\omega) = \sup_{\tau\in\mathcal{T},\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z_\tau$.

Given $(\tau,\mathbb{P}) \in \mathcal{T} \times \mathcal{P}(\Omega)$, we find $\bar{\mathbb{P}} \in \mathcal{P}(\bar{\Omega})$ so that $\mathbb{E}^{\mathbb{P}} Z_\tau = \mathbb{E}^{\bar{\mathbb{P}}} Z$.
So $\mathcal{P}(\bar{\Omega})$ contains the "convex hull" of $\mathcal{T} \times \mathcal{P}(\Omega)$.

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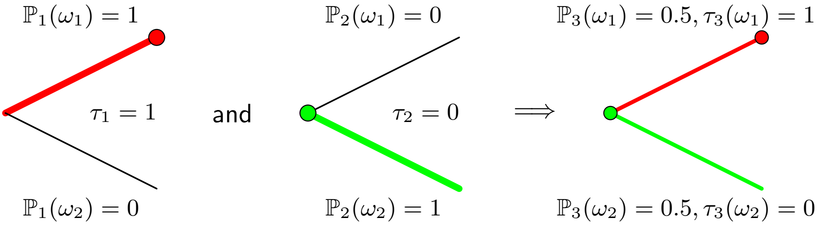
But $\bar{\mathbb{P}}\in\mathcal{P}(\bar{\Omega})$ in general may correspond to "proper" random times, not even randomised stopping times.



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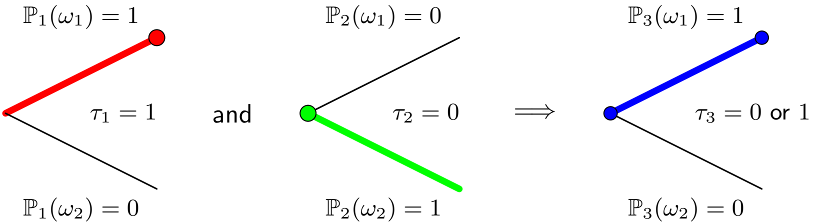

We fix this by "redistributing the stopped branches".

From random times to stopping times

Definition (Randomised stopping time)

A *randomised stopping time* corresponds to an adapted, right-continuous, increasing process A with $A_0 = 0$ and $A_1 = 1$, representing the proportion that has not been stopped.

Given a payoff function ψ, the expected payoff of a randomised stopping time is

$$\mathbb{E} \int_0^1 \psi_t \, dA_t.$$

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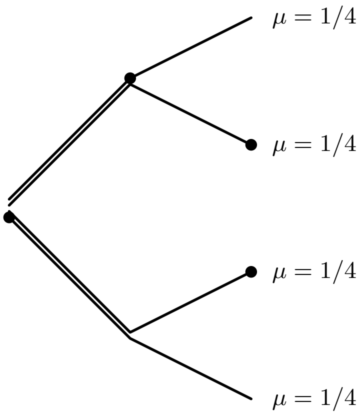
Lemma

For every $\mu \in \mathcal{P}(\bar{\Omega})$, there exists a randomised stopping time A and $\mathbb{P} \in \mathcal{P}(\Omega)$ with $\mathbb{P} \ll \mu_\Omega$, such that for every (non-anticipative) $\psi \in L^\infty(\Lambda)$,

$$\mu(\psi(\theta, \omega_{\cdot \wedge \theta})) = \mathbb{E}^{\mathbb{P}} \int_0^1 \psi(t, \omega_{\cdot \wedge t}) \, dA_t.$$

Redistributing stopped branches



Redistributing stopped branches

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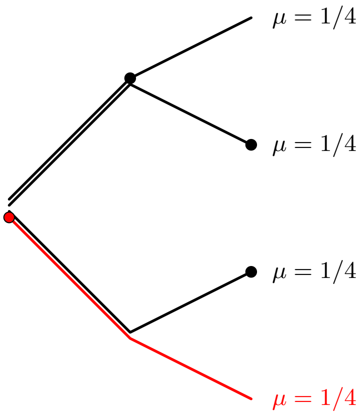

Redistributing stopped branches

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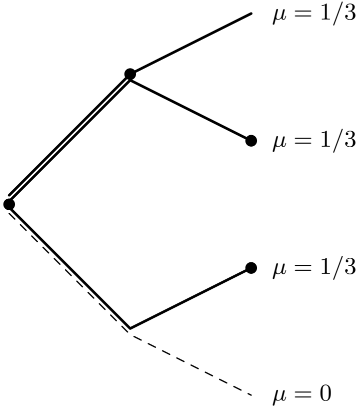

Redistributing stopped branches

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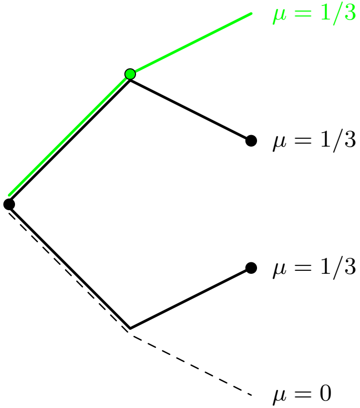

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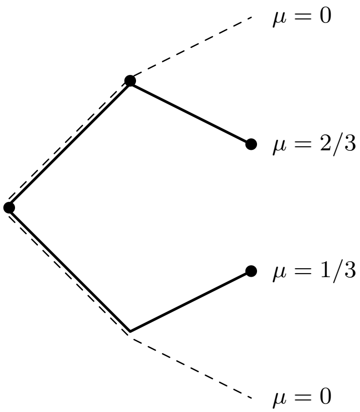

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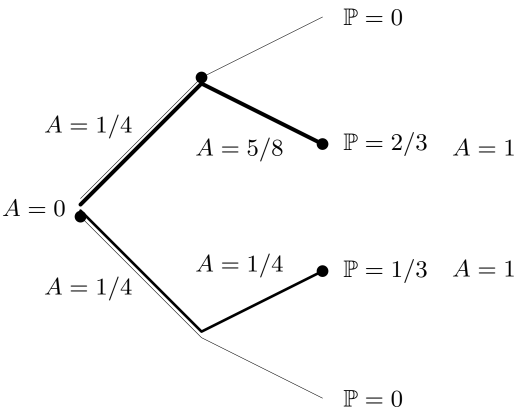

Preservation of the martingale property

Lemma

Suppose that X is a martingale under $\mu \in \mathcal{P}(\bar{\Omega})$ with characteristic $(0, \beta)$. Then there exists a family of true stopping times τ_r and $\mathbb{P} \in \mathcal{P}(\Omega)$ with characteristic $(0, \beta(t, t, \omega_{\cdot\wedge t}))$ such that for every (non-anticipative) $\psi \in L^\infty(\Lambda)$,

$$\mu(\psi(\theta, \omega_{\cdot\wedge\theta})) = \mathbb{E}^{\mathbb{P}} \int_0^1 \psi(t, \omega_{\cdot\wedge t})\, dA_t = \int_0^1 \mathbb{E}^{\mathbb{P}} \psi(\tau_r, \omega_{\cdot\wedge\tau_r})\, dr.$$

In particular, X is a martingale under $\mathbb{P}$.

In some sense, when testing against non-anticipative functions ψ, the extremal elements of $\bar{\mathcal{Q}}$ are in $\mathcal{T} \times \mathcal{Q}$.

Theorem

For any $Z \in L^0(\Lambda)$ bounded from below,

$$\sup_{\tau\in\mathcal{T}, \mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z_\tau = \sup_{\bar{\mathbb{P}}\in\bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z(\theta, \omega).$$

Disintegrating martingale measures.

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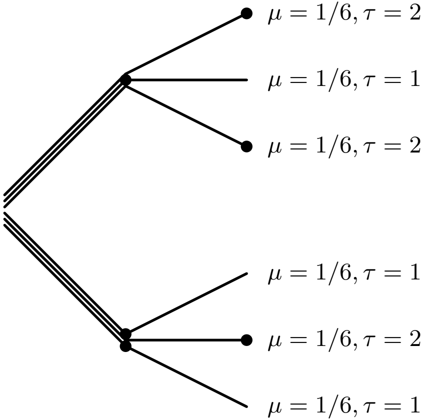

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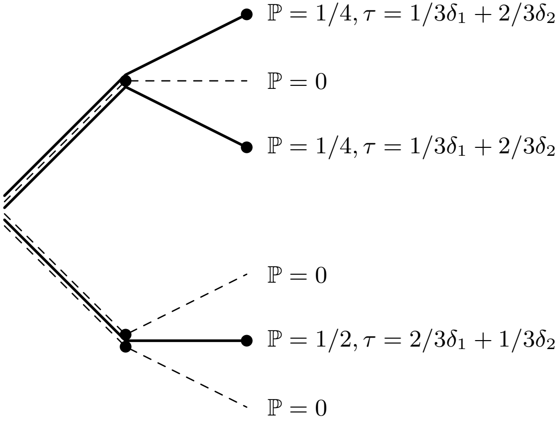

Pricing hedging duality for American options

Combining everything, we have the first main result.

Theorem

Suppose $Z \in C_b(\Lambda)$, *then*

$$\bar{\pi}(Z) \geq \pi^A(Z) \geq \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z_\tau = \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \bar{\pi}(Z).$$

Hence there's equality throughout, so

$$\pi^A(Z) = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z_\tau.$$

Suppose the market also has some statically traded European options g, which WLOG have initial prices of 0. We are allowed to trade X dynamically and g statically.

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Consider some other European claim Z.

The worst case model price is now taken over the set of calibrated models $\mathcal{Q}_g = \{\mathbb{P} \in \mathcal{Q} : \mathbb{E}^{\mathbb{P}} g = 0\}$,

$$\sup_{\mathbb{P} \in \mathcal{Q}_g} \mathbb{E}^{\mathbb{P}} Z.$$

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$$\sup_{\mathbb{P} \in \mathcal{Q}_g} \mathbb{E}^{\mathbb{P}} Z.$$

The super-hedging price includes the static trading of g

$$\pi_g(Z) := \inf \left\{ x : \exists (q,h), \text{ s.t. } x + \int_0^1 q \cdot dX_t + h \cdot g \geq Z, \ \mathcal{Q}_g\text{-q.s.} \right\}.$$

Once again, by using optimal transport duality, we have

$$\sup_{\mathbb{P} \in \mathcal{Q}_g} \mathbb{E}^{\mathbb{P}} Z = \pi_g(Z)$$

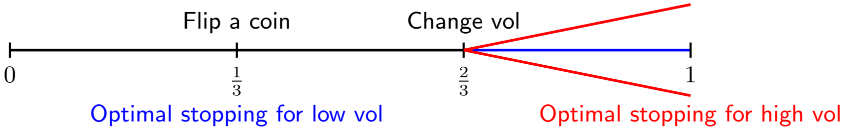
But for an American claim Z, there could be a duality gap!

$$\pi_g^A(Z) > \sup_{\mathbb{P}\in\mathcal{Q},\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}} Z_\tau$$

Counterexample: Flip a coin at $t = 1/3$ which decides the volatility for $t \geq 2/3$.

Consider an American payoff where it's optimal to stop at $t = 1$ if the volatility is high, or $t = 1/3$ if the volatility is low. Then knowledge of the coin flip can improve the optimal stopping.

$$\pi_g^A(Z) = \sup_{\widehat{\mathbb{P}}\in\widehat{\mathcal{Q}},\widehat{\tau}\in\widehat{\mathcal{T}}} \mathbb{E}^{\widehat{\mathbb{P}}} \widehat{Z}_{\widehat{\tau}} > \sup_{\mathbb{P}\in\mathcal{Q},\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}} Z_\tau$$



We recover duality by embedding the dynamic prices of g into a larger space $\widehat{\Omega}$, and allowing stopping times to see these prices.

Theorem

Let $Z \in C_b(\Lambda)$ *and* $g \in C_b(\mathbb{R}^m)$. *Then*

$$\bar{\pi}_g(\bar{Z}) \geq \pi_g^A(Z) \geq \widehat{\pi}^A(\widehat{Z}) \geq \sup_{\widehat{\mathbb{P}} \in \widehat{\mathcal{Q}}, \widehat{\tau} \in \widehat{\mathcal{T}}} \mathbb{E}^{\widehat{\mathbb{P}}} \widehat{Z}_{\widehat{\tau}} = \sup_{\bar{\widehat{\mathbb{P}}} \in \bar{\widehat{\mathcal{Q}}}} \mathbb{E}^{\bar{\widehat{\mathbb{P}}}} \bar{\widehat{Z}} \geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}_g} \mathbb{E}^{\bar{\mathbb{P}}} \bar{Z} = \bar{\pi}_g(\bar{Z}).$$

Therefore, there's equality throughout,

$$\pi_g^A(Z) = \sup_{\widehat{\mathbb{P}} \in \widehat{\mathcal{Q}}, \widehat{\tau} \in \widehat{\mathcal{T}}} \mathbb{E}^{\widehat{\mathbb{P}}} \widehat{Z}_{\widehat{\tau}}.$$

Interestingly, the robust super-hedge of Z does not care about the dynamic prices of g, nor does it need to trade g dynamically.

We recover duality by embedding the dynamic prices of g into a larger space $\widehat{\Omega}$, and allowing stopping times to see these prices.

Assume: $X_0 = x_0 \in \mathbb{R}^d$, Diff. Char. $\in \mathfrak{E}(t, \omega_{\cdot\wedge t}) \subseteq \mathbb{S}^d_+$ closed, convex, and globally bounded for $(t, \omega_{\cdot\wedge t}) \in \Lambda$ and such that $\mathfrak{E}$ is continuous with respect to the topology induced by the Hausdorff metric on compact subsets.

Theorem

Let $Z \in C_b(\Lambda)$ and $g \in C_b(\mathbb{R}^m)$. Then

$$\bar{\pi}_g(\bar{Z}) \geq \pi^A_g(Z) \geq \widehat{\pi}^A(\widehat{Z}) \geq \sup_{\widehat{\mathbb{P}} \in \widehat{\mathcal{Q}}, \widehat{\tau} \in \widehat{\mathcal{T}}} \mathbb{E}^{\widehat{\mathbb{P}}} \widehat{Z}_{\widehat{\tau}} = \sup_{\bar{\widehat{\mathbb{P}}} \in \bar{\widehat{\mathcal{Q}}}} \mathbb{E}^{\bar{\widehat{\mathbb{P}}}} \bar{\widehat{Z}} \geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}_g} \mathbb{E}^{\bar{\mathbb{P}}} \bar{Z} = \bar{\pi}_g(\bar{Z}).$$

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Interestingly, the robust super-hedge of Z does not care about the dynamic prices of g, nor does it need to trade g dynamically.

Thank You

papers and more available at
`http://people.maths.ox.ac.uk/obloj/`.

Let $\widehat{\Omega} = D([-\delta, 1]; \mathbb{R}^{d+m})$ with the canonical process $\widehat{X} = (X, Y)$.
Given $E_1 \in \mathcal{B}(\Omega)$ and $E_2 \in \mathcal{B}(\Lambda)$ we let

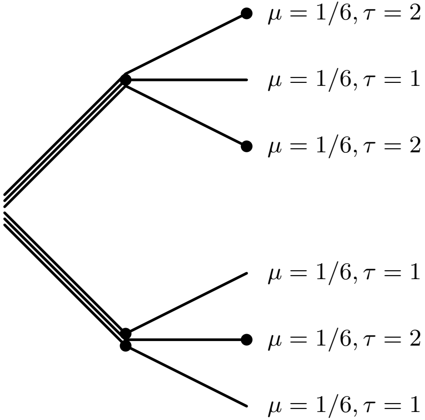
$$\begin{aligned}\widehat{E}_1 &:= \{\widehat{\omega} \in \widehat{\Omega} : \omega_{|[0,1]} \in E_1,\ X_t(\widehat{\omega}) = X_0(\widehat{\omega}), Y_t(\widehat{\omega}) = 0, t \in [-\delta, 0),\\ &\qquad\qquad Y_1(\widehat{\omega}) = g(X(\widehat{\omega}))\} \in \mathcal{B}(\widehat{\Omega}),\\ \widehat{E}_2 &:= \Big\{(t, X_{\cdot\wedge t}, Y_{\cdot\wedge t}) \in \widehat{\Lambda} : (t, X_{\cdot\wedge t})_{t\in[0,1]} \in E_2\Big\} \in \mathcal{B}(\widehat{\Lambda}).\\ \widehat{\mathcal{Q}}^{\widehat{E}} &= \{\widehat{\mathbb{P}} \in \widehat{\mathcal{Q}} : \widehat{\mathbb{P}}(\widehat{E}_1) = \lambda \otimes \widehat{\mathbb{P}}(\widehat{E}_2) = 1\}.\end{aligned}$$

Lemma: every model in $\mathcal{Q}^E_g$ can be lifted to a model in $\widehat{\mathcal{Q}}^{\widehat{E}}$.

Lemma

For any $\widehat{\mathbb{P}} \in \widehat{\mathcal{Q}}^{\widehat{E}}$, $\widehat{\mathbb{P}} \circ \big((X_t)_{t\in[0,1]}\big)^{-1} \in \mathcal{Q}^E_g$. Conversely, for any $\mathbb{P} \in \mathcal{Q}^E_g$, there exists a $\widehat{\mathbb{P}} \in \widehat{\mathcal{Q}}^{\widehat{E}}$ such that the $\widehat{\mathbb{P}} \circ \big((X_t)_{t\in[0,1]}\big)^{-1} = \mathbb{P}$.

Disintegrating martingale measures.



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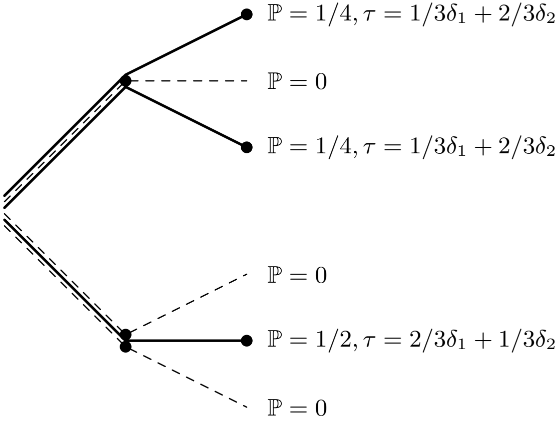

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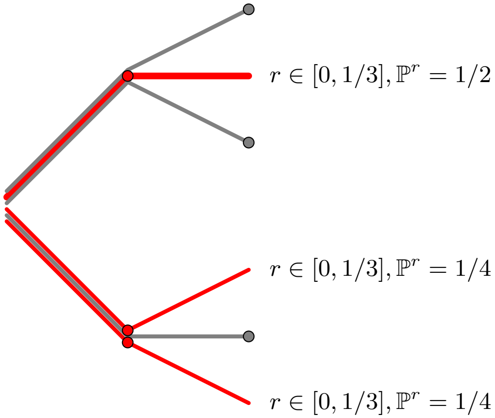

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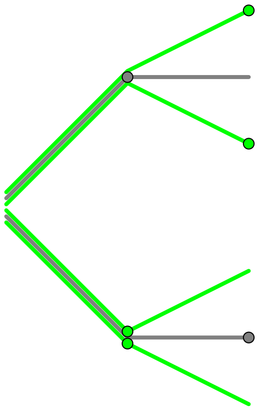

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