Efficient Simulation of Affine Volterra Processes: From Heston to Hawkes

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Based on joint works with
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Philosophy of the talk

- Financial markets exhibit **spiky behavior**: sudden, large moves on short horizons.
- **Continuous** models can reproduce this via **extreme parameter** regimes (fast mean reversion, large vol-of-vol, roughness, etc...).
- **Naturally**, numerically, these regimes favor **jump-based schemes** over Brownian diffusive discretizations.

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Motto

If it looks like a jump, simulate it like a jump!

The Heston Model

The Heston (1993) model is one of the **most popular** stochastic volatility models (in academia):

$$dS_t = S_t \sqrt{V_t} dB_t,$$
$$dV_t = (a + bV_t)dt + c\sqrt{V_t} dW_t,$$

where $B_t = \rho W_t + \sqrt{1-\rho^2} W_t^{\perp}$, and parameters satisfy

$$V_0, a, c \geq 0, \quad b \leq 0, \quad \rho \in [-1, 1].$$

Why so Popular?

The Heston model is widely used due to its **affine structure**

- **Tractability** The characteristic function admits a closed-form solution, allowing fast pricing, hedging and calibration by Fourier-based methods.

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The joint conditional characteristic function of $(\log S, \int V)$ is given by

$$\mathbb{E}\left[\exp\left(v\log\frac{S_T}{S_t} + w\int_t^T V_s ds\right)\middle| V_t\right] = \exp\left(\phi\left(T-t\right) + \psi\left(T-t\right)V_t\right), \quad v, w \in i\mathbb{R},$$

where (ϕ, ψ) solve the following system of **Riccati equations**

$$\phi'(t) = a\psi(t), \quad \phi(0) = 0,$$
$$\psi'(t) = \frac{c^2}{2}\psi^2(t) + (\rho c v + b)\psi(t) + w + \frac{v^2 - v}{2}, \quad \psi(0) = 0.$$

BUT

. . . the **affine structure** also introduces challenges

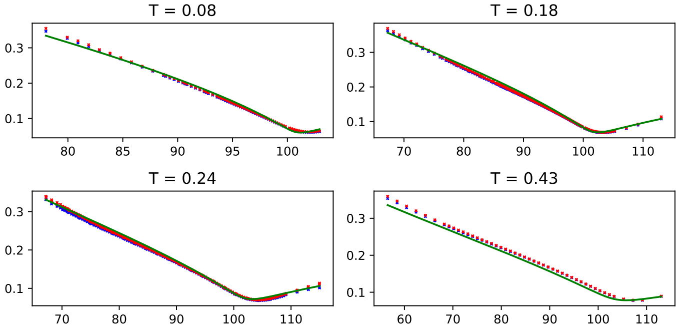
- Square-root term in the diffusion makes **simulation challenging**, especially for realistic calibrated parameters, extreme vol-of-vol c and mean reversion b.
- The process V will spend too much time **near zero**.
- Non-negativity must be preserved, complicating discretization.

$$dS_t = S_t\sqrt{V_t}dB_t, \quad dV_t = (a + bV_t)dt + c\sqrt{V_t}dW_t,$$

The Heston Model

Typical calibration

V_0	a	b	c	ρ
0.006	17.25×0.018	-17.25	2.95	-0.68



Implied volatility surface of the calibrated Heston model (green) on the market bid and ask implied volatilities (red and blue dots) on **October 10, 2017**.

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This motivates two key questions

- **Mathematical** Can we reconcile the affine structure with the process dynamics for simulation?
- **Practical** Can we design a scheme that performs well in challenging market regimes with very few discretization steps?

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- **Practical** Can we design a scheme that performs well in challenging market regimes with very few discretization steps?

$\Rightarrow$ Need for a **new perspective** . . .

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Introduce the **integrated quantities**

$$U_{s,t} := \int_s^t V_r\, dr, \quad Z_{s,t} := \int_s^t \sqrt{V_r}\, dW_r, \quad s \le t.$$

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so that for a partition $0 = t_0 < t_1 < \ldots < t_n = T$:

$$S_{t_{i+1}} = S_{t_i} \exp\left(-\frac{1}{2}U_{t_i,t_{i+1}} + \rho Z_{t_i,t_{i+1}} + \sqrt{1-\rho^2}\int_{t_i}^{t_{i+1}} \sqrt{V_s}dW_s^{\perp}\right)$$

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Conditional on $(V_s)_{t_i \leq s \leq t_{i+1}}$,

$$\int_{t_i}^{t_{i+1}}\sqrt{V_s}dW_s^{\perp} \sim \sqrt{U_{t_i,t_{i+1}}}\mathcal{N}(0,1).$$

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(!) To sample S it suffices to sample (U, Z).

Conventional Perspective

- There is a **substantial** body of literature on the simulation of square-root processes and the Heston model
- The approaches in the literature generally follow these **four main steps**:
  - **Step 1 Update $V_{t_{i+1}}$ given V_{t_i}.**
  - **Step 2 Sample $U_{t_i,t_{i+1}}$ knowing $(V_{t_i}, V_{t_{i+1}})$.**
  - **Step 3 Deduce $Z_{t_i,t_{i+1}}$ using the dynamics of V**

$$Z_{t_i,t_{i+1}} = \frac{1}{c}\left(V_{t_{i+1}} - V_{t_i} - a(t_{i+1} - t_i) - bU_{t_i,t_{i+1}}\right).$$

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    - $\Rightarrow$ Approximate conditional distribution using moment matching: Inverse Gaussian Tse and Wan (2013), Inverse Gamma Bégin, Bédard, and Gaillardetz (2015), Poisson conditioning Choi and Kwok (2024)...

Some methods combine both dynamics-based and distribution-based steps as the **Quadratic Exponential (QE)** scheme of Andersen (2007):

- **For Step 1** Switching mechanism between squared Gaussian and tweaked exponential distributions for $V_{t_{i+1}}$.
- **For Step 2** Approximates $U_{t_i,t_{i+1}}$ via the mid-point rule $(V_{t_i} + V_{t_{i+1}})/2$.

Despite its hybrid nature, QE remains structured within the conventional steps.

It is widely used in practice and serves as a benchmark for accuracy and efficiency.

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To simulate V we start by sampling V.

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But Why?

The iVi scheme

Based on

- *Simulation of square-root processes made simple: applications to the Heston model*, **Risk Magazine (Cutting Edge section)** (2025).

Sample the **integrated variance** $U_{t_i,t_{i+1}} = \int_{t_i}^{t_{i+1}} V_s dr$ first

The new perspective builds upon integrated quantities

$$U_{s,t} = \int_s^t V_r \, dr, \quad Z_{s,t} = \int_s^t \sqrt{V_r} \, dW_r, \quad s \leq t.$$

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Dynamics of the integrated variance

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A right endpoint rule leads to

$$U_{t_i,t_{i+1}} \approx \alpha_i + b\Delta t U_{t_i,t_{i+1}} + c\Delta t Z_{t_i,t_{i+1}}$$

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This is an **implicit** equation on U:

- **Affine structure:** $Z_{t_i,t_{i+1}} = \int_{t_i}^{t_i+1} \sqrt{V_s} dW_s$ has a quadratic variation that is equal to U.
- **Dambis, Dubins-Schwarz Theorem** yields $Z_{t_i,t_{t_{i+1}}} = \widetilde{W}_{U_{t_i,t_{i+1}}}$, with $\widetilde{W}$ a Brownian motion.

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Hence, approximating $U_{t_i,t_{i+1}}$ boils down to finding a nonnegative rv $\widehat{U}_{i,i+1}$ solving

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Said differently $\widehat{U}_{i,i+1}$ is a passage time of the level α_i for the drifted Brownian motion $((1 - b\Delta t)s - c\Delta t \widetilde{W}_s)_{s\geq 0}$.

In particular, the first passage time

$$X = \inf\left\{s \geq 0 : (1 - b\Delta t)s - c\Delta t\widetilde{W}_s = \alpha_i\right\}$$

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It is well known that X follows an **Inverse Gaussian distribution** $IG(\mu_i, \lambda_i)$ with mean parameter $\mu_i = \alpha_i/(1 - b\Delta t)$ and shape parameter $\lambda_i = \frac{{\alpha_i}^2}{(c\Delta t)^2}$.

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Its characteristic function is given by

$$\mathbb{E}\left[\exp\left(wX\right)\right] = \exp\left(\frac{\lambda_i}{\mu_i}\left(1 - \sqrt{1 - \frac{2w\mu_i^2}{\lambda_i}}\right)\right),$$

for all $w \in \mathbb{C}$ such that $\mathfrak{Re}(w) \leq 0$.

PROBABILITÉ POUR QU'UN COURS SOIT ATTEINT DANS UN INTERVALLE DE TEMPS DONNÉ.

Cherchons la probabilité P pour qu'un cours donné c soit atteint ou dépassé dans un intervalle de temps t.

Supposons d'abord, pour simplifier, que le temps soit décomposé en deux unités, que t égale deux jours par exemple.

Soit x le cours coté le premier jour et soit y le cours du second jour relativement à celui du premier.

Louis Bachelier, Théorie de la spéculation 1900, p.70 https://www.numdam.org/article/ASENS_1900_3_17__21_0.pdf

- IG distribution first derived by Bachelier (1900) as the time a stock first reaches a certain price.
- Schrödinger (1915) used it for the first-passage time of Brownian motion.
- The term "Inverse Gaussian" was proposed by Tweedie (1945).

IG random variables can be simulated easily using one Gaussian and one Uniform random variable using an acceptance-rejection step as shown in Michael, Schucany, and Haas (1976)

Algorithm Sampling from the Inverse Gaussian Distribution with Parameters $\mu > 0$, $\lambda > 0$

1: Generate $\xi \sim \mathcal{N}(0,1)$ and compute $Y = \xi^2$.
2: Compute the candidate value X:

$$X = \mu + \frac{\mu^2 Y}{2\lambda} - \frac{\mu}{2\lambda}\sqrt{4\mu\lambda Y + \mu^2 Y^2}.$$

3: Generate a uniform random variable: Sample $\eta \sim \text{Uniform}(0,1)$.
4: **if** $\eta \leq \frac{\mu}{\mu+X}$ **then**
5: Set the output $IG = X$.
6: **else**
7: Set $IG = \frac{\mu^2}{X}$.
8: **end if**

Leads to **integrated Variance implicit (iVi)** scheme

This leads to the **integrated Variance implicit (iVi)** scheme

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Initialize $\alpha_0 = V_0 \Delta t + a\frac{(\Delta t)^2}{2}$.

- Step 1: knowing α_i:

$$\widehat{U}_{i,i+1} \sim IG\left(\frac{\alpha_i}{1 - b\Delta t}, \frac{\alpha_i{}^2}{(c\Delta t)^2}\right)$$

- Step 2: Set

$$\widehat{Z}_{i,i+1} = \frac{1}{c\Delta t}\left((1 - b\Delta t)\widehat{U}_{i,i+1} - \alpha_i\right)$$

- Step 3 Update $\widehat{V}$ and α:

$$\widehat{V}_{i+1} = \widehat{V}_i + a\Delta t + b\widehat{U}_{i,i+1} + c\widehat{Z}_{i,i+1}$$

$$\alpha_{i+1} = \widehat{V}_{i+1}\Delta t + a\frac{(\Delta t)^2}{2}.$$

$$\mathbb{E}\left[e^{zU_{t_i,t_{i+1}}} \middle| V_{t_i}\right] = e^{\phi(\Delta t)+\psi(\Delta t)V_{t_i}}$$

where (ϕ, ψ) solve the following system of Riccati equations:

$$\phi'(t) = a\psi(t), \quad \phi(0) = 0, \quad \psi'(t) = \frac{c^2}{2}\psi^2(t) + b\psi(t) + z, \quad \psi(0) = 0.$$

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An (implicit) right endpoint rule on the Riccati equation yields

$$\psi(\Delta t) \approx z\Delta t + b\Delta t\psi(\Delta t) + \Delta t\frac{c^2}{2}\psi^2(\Delta t)$$

so that

$$\widehat{\psi} = \frac{(1-b\Delta t) - \sqrt{(1-b\Delta t)^2 - 2z(\Delta t)^2 c^2}}{c^2\Delta t}, \quad \widehat{\phi} = a\Delta t\widehat{\psi}.$$

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Since $\widehat{U}_{i,i+1}\sim IG\left(\frac{\alpha_i}{1-b\Delta t},\frac{{\alpha_i}^2}{(c\Delta t)^2}\right)$, using the characteristic function of IG:

$$\mathbb{E}\left[e^{z\widehat{U}_{i,i+1}}\middle|\alpha_i\right]=e^{\frac{\alpha_i}{\Delta t}\widehat{\psi}}=e^{\widehat{\phi}+\widehat{\psi}V_i}\approx\mathbb{E}\left[e^{zU_{t_i,t_{i+1}}}\middle|V_{t_i}\right]$$

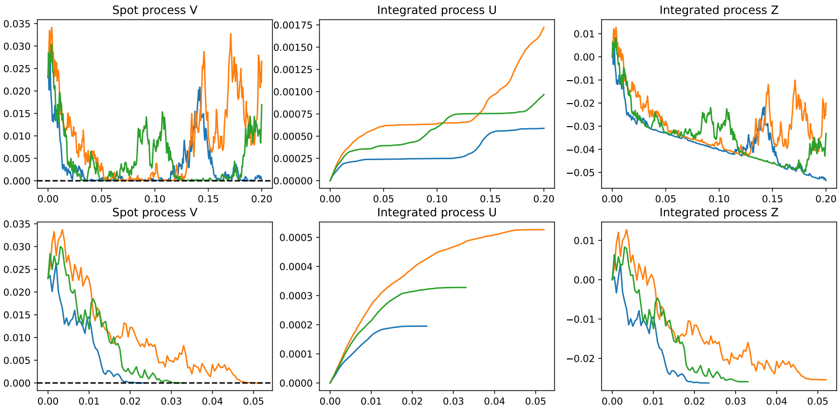
Theorem

Let $V_0, a, c \geq 0$ and $b \leq 0$. Consider $(\widehat{V}_i)_{i=0,\ldots,n}$, $(\widehat{U}_{i,i+1})_{i=0,\ldots,n-1}$, and $(\widehat{Z}_{i,i+1})_{i=0,\ldots,n-1}$ satisfying the recursions of the iVi scheme. Then, we have that

$$\widehat{V}_i, \alpha_i \geq 0, \quad i = 0, \ldots, n.$$

Remark

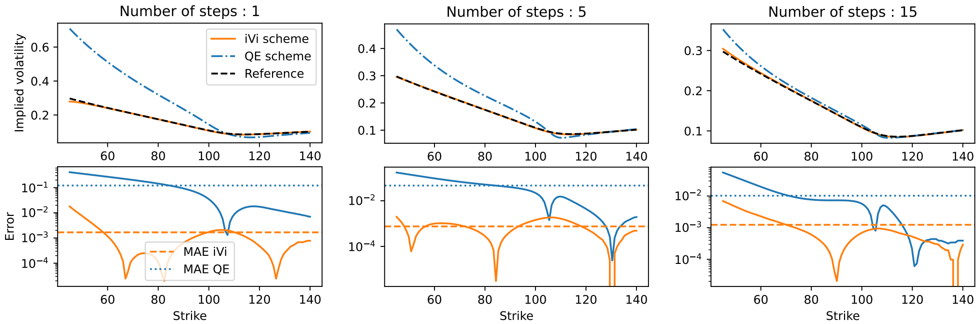
In practice, variation of constants to reduce bias then apply the iVi scheme.



Row 1 - Reflecting boundary $a > 0$: Parameters as in Case 2 in Table ?? below. Row 2 - absorbing boundary: $a = 0$ and other parameters unchanged. $T = 0.2$ and a 200 time steps.

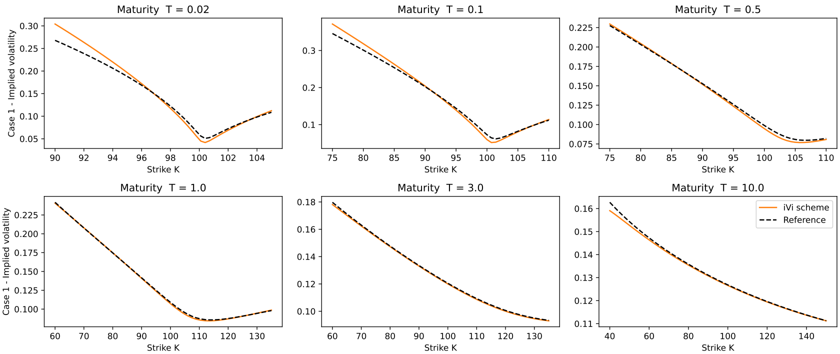
V_0	a	b	c	ρ
0.006	17.25×0.018	-17.25	2.95	-0.68

- calibrated on S&P500 options up to 6 months
- Reference values: Fourier using Lewis formula.
- For comparison: QE scheme of Andersen (2007).



Case 1: Implied volatility slice for $T = 1$ and 2 million sample paths.

Full surface iVi scheme (in orange) is computed with only **one single time step** (equal to the maturity T) per slice!



Heston's Implied volatility surface with the iVi scheme with one single time step and 2 million sample paths for model parameters.

Why?

- Under the parameterization $c = -b\beta$ and $a = -b\gamma$, with $\beta, \gamma > 0$, as $b \to \infty$, the distribution of $U_{0,T}$ converges to an Inverse Gaussian distribution, see Mechkov (2015), and AJ and De Carvalho (2024); McCrickerd (2019).
- For large maturities, independently of the parameters, the limiting distribution of $U_{0,T}$ as $T \to \infty$ is also an Inverse Gaussian distribution as shown by Forde and Jacquier (2011).

Extending **iVi** to the **Volterra World**

Based on

- *Simulating integrated Volterra square-root processes and Volterra Heston models via Inverse Gaussian*, with **Elie Attal** (2025).

https://sites.google.com/view/abijabereduardo/

Volterra Heston mode, defined by

$$dS_t = S_t\sqrt{V_t}dB_t, \quad d\langle B, W\rangle_t = \rho dt$$

$$V_t = g_0(t) + b\int_0^t K(t-s)V_s\, ds + c\int_0^t K(t-s)\sqrt{V_s}\, dW_s,$$

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- with $K \in L^2([0,T],\mathbb{R})$
  - standard Heston (1993) model $K(t) = 1$ or $K(t) = e^{-\lambda t}$
  - Lifted Heston model AJ (2019)

$$K(t) = \sum_{i=1}^{n} c_i e^{-\lambda_i t}.$$

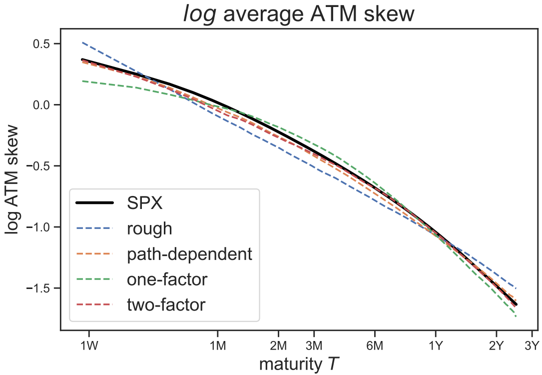
  - Rough Heston El Euch and Rosenbaum (2019)

$$K(t) = ct^{H-1/2}, \quad H \in (0, 1/2)$$

  - Shifted fractional kernel

$$K(t) = c(\epsilon + t)^{H-1/2}, \quad \epsilon > 0, \quad H \in \mathbb{R}.$$

Log-plot SPX ATM skew is **concave**, **flattening behavior** at short maturities



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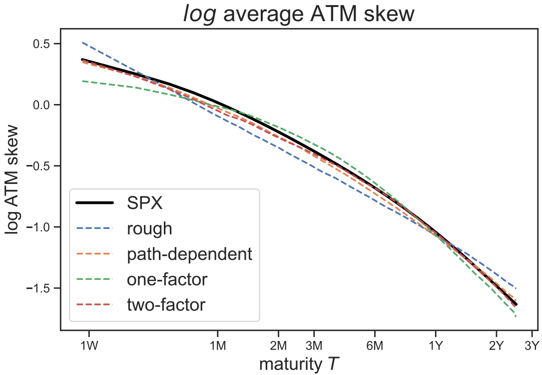

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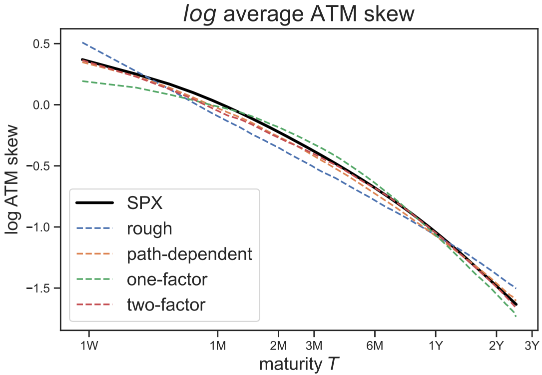

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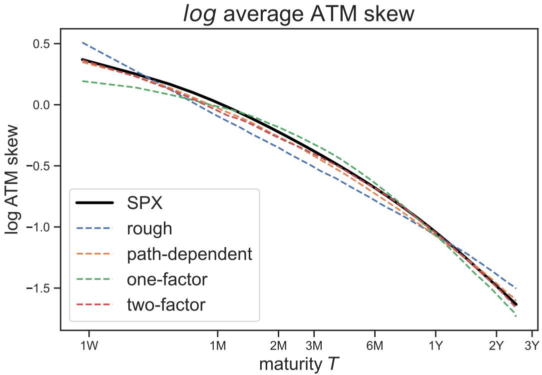

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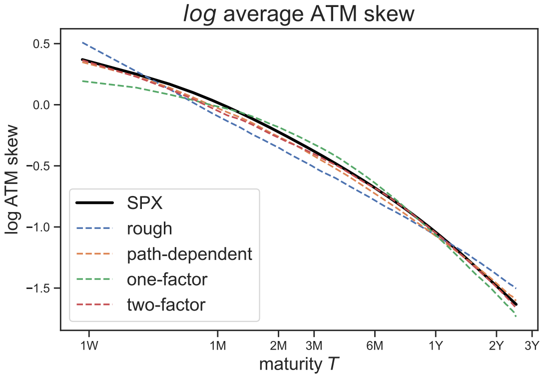

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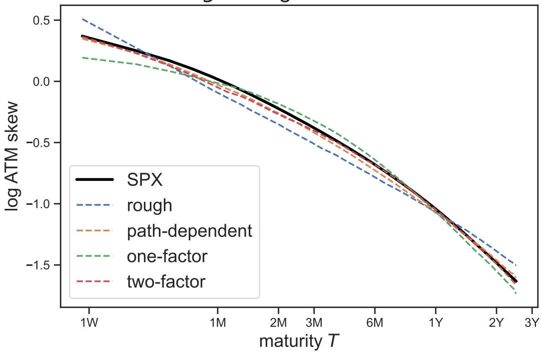
Log-plot SPX ATM skew is **concave**, **flattening behavior** at short maturities

log average ATM skew



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- AJ and Li (2025); Bergomi (2015); Delemotte, Marco, and Segonne (2023); Guyon and El Amrani (2022)

Existing approaches for simulation of Volterra Heston based on

- **Multifactor approximations** AJ (2019); AJ and El Euch (2019); Alfonsi and Kebaier (2024); Bayer and Breneis (2024)...
- **Explicit Euler type** Alfonsi (2025); Richard, Tan, and Yang (2023) ...
- **QE extension** Gatheral (2022)

typically struggle with spiky regimes (high mean reversions, low Hurst index H, etc...)

Dynamics of the integrated variance

$$U_{0,t} := \int_0^t V_s ds.$$

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Stochastic Fubini theorem yields the dynamics

$$U_{0,t} = \int_0^t g_0(s)\, ds + \int_0^t K(t-s)\,(bU_{0,s} + cZ_{0,s})\, ds,$$

where $Z_{0,\cdot} := \int_0^\cdot \sqrt{V_s}\, dW_s$ is a continuous martingale with quadratic variation U.

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- **Advantage** the equation remains valid for kernels K that are only locally integrable, but not necessarily locally square-integrable, see AJ (2021).
- For instance, it holds for the fractional kernel K_H, even for negative Hurst indices $H \in (-1/2, 1/2]$, see Jusselin and Rosenbaum (2020).

Write the dynamics of the increments $U_{s,t} = U_{0,t} - U_{0,s}$.
For this, we define, for $s \leq t$,

$$G_s(t) = \int_s^t g_0(u)\,du + \int_0^s \int_s^t K(u-r)\,du\,(b\,dU_{0,r} + c\,dZ_{0,r})$$

and note that $G_s(t)$ is $\mathcal{F}_s$-measurable for each $t \geq s$.

Theorem

Dynamics of U

$$U_{s,t} = \underbrace{G_s(t)}_{\text{summarizing the non-Markovianity}} + \int_s^t K(t-r)\,(bU_{s,r} + cZ_{s,r})\,dr, \quad s \leq t.$$

Theorem

$$\mathbb{E}\left[\exp\left(wU_{t_i,t_{i+1}}\right) \middle| \mathcal{F}_{t_i}\right] = \exp\left(\int_{t_i}^{t_{i+1}} F(\psi(t_{i+1}-s))dG_{t_i}(s)\right),$$

where ψ is the solution of the following Riccati Volterra equation

$$\psi(t) = \int_0^t K(t-s)F(\psi(s))ds, \quad F(u) = w + bu + \frac{c^2}{2}u^2.$$

Refer to AJ (2021) for the case of L^1_{loc} kernels, and AJ, Larsson, and Pulido (2019) for the equivalent formulations in the specific (and more standard) case of L^2_{loc} kernels.

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Refer to AJ (2021) for the case of L^1_{loc} kernels, and AJ, Larsson, and Pulido (2019) for the equivalent formulations in the specific (and more standard) case of L^2_{loc} kernels.
Right-end point between t_i and t_{i+1} on Riccati leads to **second order equation**

$$\psi(\Delta_i) \approx w\int_0^{\Delta_i} K(s)\,ds + \int_0^{\Delta_i} K(\Delta_i - s)\,ds\left(b\psi(\Delta_i) + \frac{c^2}{2}\psi^2(\Delta_i)\right),$$

which can be solved explicitly for $\psi(\Delta_u)$.

So that

$$\mathbb{E}\left[\exp\left(wU_{i,i+1}\right) \mid \alpha_i\right] = \exp\left(\int_{t_i}^{t_{i+1}} F(\psi(t_{i+1}-s))dG_{t_i}(s)\right) \approx \exp\left(F(\widehat{\psi}_{i,i+1})G_{t_i}(t_{i+1})\right)$$

which yields after discretization:

$$G_{t_i}(t_{i+1}) \approx \int_{t_i}^{t_{i+1}} g_0(s)ds + \sum_{j=0}^{i-1} k_{i-j}\left(b\widehat{U}_{j,j+1} + c\widehat{Z}_{j,j+1}\right) =: \alpha_i, \quad k_\ell = \int_0^{\frac{T}{n}} K\left(\frac{\ell T}{n} + s\right) ds,$$

that

$$\widehat{U}_{i,i+1} \sim IG\left(\frac{\alpha_i}{1-bk_0}, \left(\frac{\alpha_i}{ck_0}\right)^2\right)$$

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that

$$\widehat{U}_{i,i+1} \sim IG\left(\frac{\alpha_i}{1-bk_0}, \left(\frac{\alpha_i}{ck_0}\right)^2\right)$$

Using discetrization of **dynamics** we update

$$\widehat{Z}_{i,i+1} = \frac{1}{ck_0}\left((1-bk_0)\widehat{U}_{i,i+1} - \alpha_i\right).$$

Algorithm - The iVi scheme: Simulation of $\widehat{U}, \widehat{Z}$

1: Compute

$$k_\ell = \int_0^{\frac{T}{n}} K\left(\frac{\ell T}{n} + s\right) ds, \quad \ell = 0, \dots, n-1.$$

2: **for** $i = 0$ to $n-1$ **do**

3: Compute the quantity:

$$\alpha_i = \int_{t_i}^{t_{i+1}} g_0(s)ds + \sum_{j=0}^{i-1} k_{i-j}\left(b\widehat{U}_{j,j+1} + c\widehat{Z}_{j,j+1}\right),$$

with $\alpha_0 = \int_0^{t_1} g_0(s)ds$, for $i = 0$.

4: Simulate the increment: $\widehat{U}_{i,i+1} \sim IG\left(\frac{\alpha_i}{1-bk_0}, \left(\frac{\alpha_i}{ck_0}\right)^2\right)$.

5: Set: $\widehat{Z}_{i,i+1} = \frac{1}{ck_0}\left((1-bk_0)\widehat{U}_{i,i+1} - \alpha_i\right)$.

6: **end for**

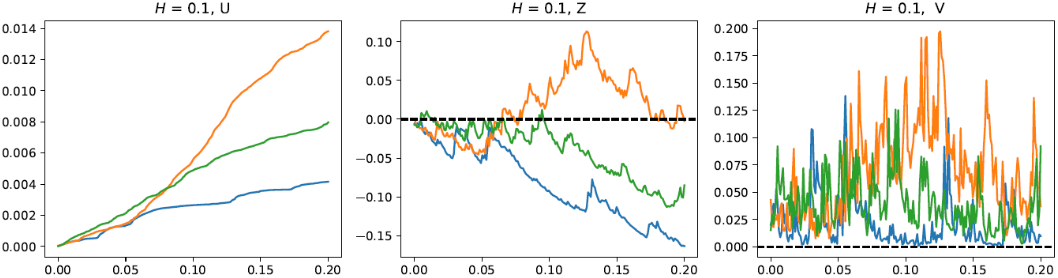
We establish:

- **Well-definedess and non-decreasing $\widehat{U}$**:
  Key ingredient the notion of kernels that preserve nonnegativity of Alfonsi (2025) but on the integrated kernel.
- **Weak convergence** in the Skorkhod topology of the scheme as the number of times steps $n \to \infty$:
  Key observation $U^n =:= \sum_{i=0}^{\lfloor nt/T \rfloor - 1} \widehat{U}^n_{i,i+1}$ solves a stochastic Volterra equation with a measure-valued kernel:

$$U^n_t = \int_0^{\lfloor \frac{nt}{T} \rfloor \frac{T}{n}} g_0(s)\, ds + \int_{[0,t]} \left(b\, U^n_{t-s} + c\, Z^n_{t-s} \right) K^n(ds) \quad t \leq T\,.$$

with $K^n(ds) := \sum_{i=0}^{n-1} k^n_i\, \delta_{t^n_i}(ds)$ on $[0\,,\,T]$.

$$K(t) = t^{H-1/2}, \quad H = 0.1$$



The parameters are $a = 0.1$, $b = -0.3$, $c = 0.2$, $V_0 = 0.04$ with a varying Hurst index $H \in \{0.1, -0.1, -0.4\}$ for each row. $T = 0.2$ and 200 time steps.

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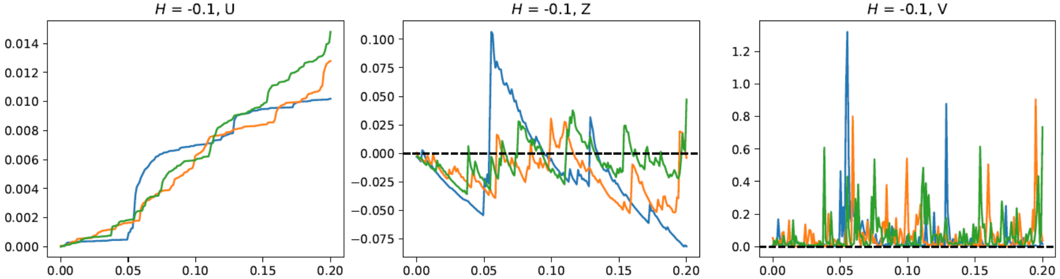

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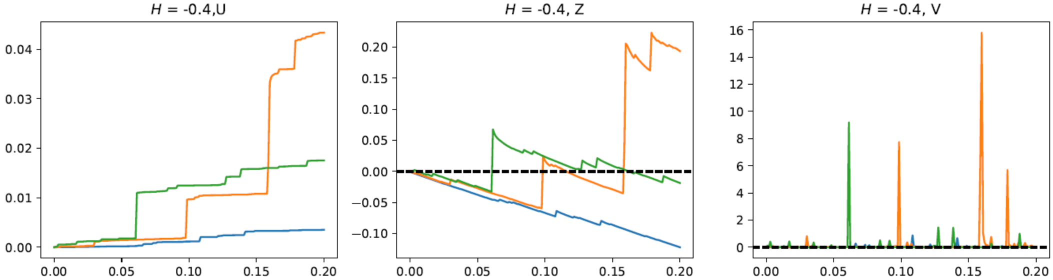
iVi for Volterra Heston

Numerics

$$K(t) = t^{H-1/2}, \quad H = -0.4$$



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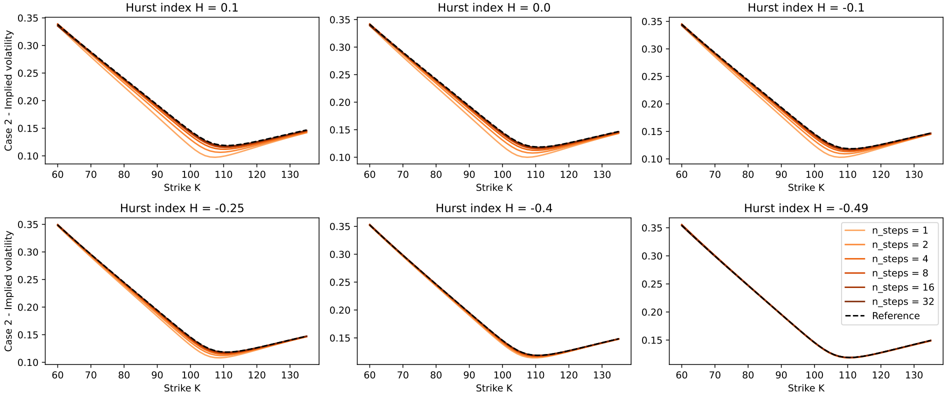
Numerics for the rough and hyper-rough case $K(t) = t^{H-1/2}$ for $H \in (-1/2, 1/2)$:



Implied volatility slice of the hyper-rough Heston model with maturity $T = 1$, with varying Hurst index between 0.1 and -0.49.

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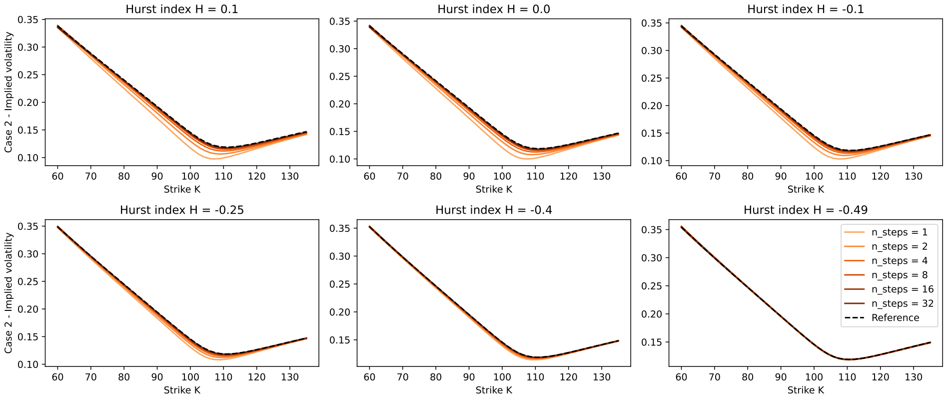


Implied volatility slice of the hyper-rough Heston model with maturity $T = 1$, with varying Hurst index between 0.1 and -0.49.

Unlike all other schemes, iVi accuracy improves as H decreases?

From Hyper Roughness to Jumps as $H \to -1/2$.

$$K_H(t) := (H + 1/2)\, t^{H-1/2}, \quad t \leq T, \quad H > -1/2.$$

Sequence of hyper-rough integrated square-root process:

$$X_t^H = G_0^H(t) + \int_0^t K^H(t-s) W_{X_s^H}\, ds$$

As $H \to -1/2$, we have $K^H \to \delta_0$ and hence limiting equation becomes IG:

$$X_t = G_0(t) + W_{X_t}$$

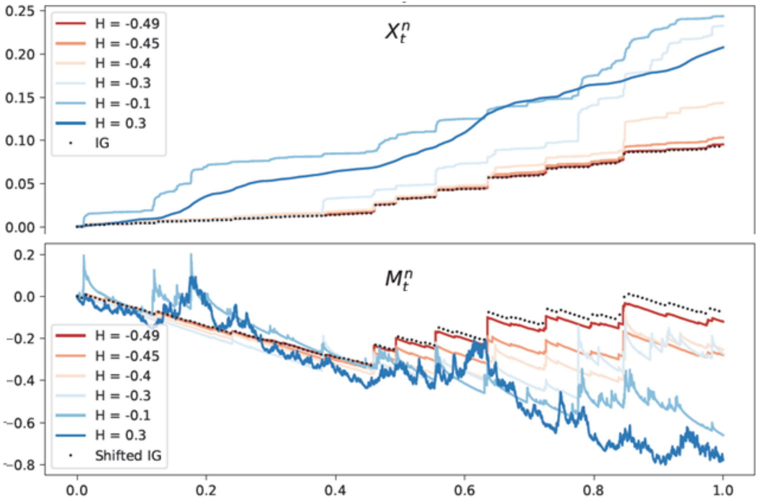
To make this rigorous - Skorokhod M_1 topology for X^H and non-Skorokhod S-topology for W_{X^H}:

Abi Jaber, E., Attal, E., & Rosenbaum, M. (2025). From Hyper Roughness to Jumps as H→-1/2.

iVi for Volterra Heston

Rough to jumps

Thanks to iVi scheme we can **easily visualize** this convergence from roughness to jumps



iVi for Hawkes processes

Based on

- *Efficient Simulation of Hawkes Processes using their Affine Volterra Structure*, with **Elie Attal** and **Dimitri Sotnikov** (2025).

Hawkes (1971) process: counting process $(N_t)_{t\geq 0}$ with intensity

$$\lambda_t = g_0(t) + \int_0^t K(t-s)\, dN_s,$$

with g_0 the exogenous intensity, and the convolution term with $K \in L^1$ is the endogenous (self-exciting) part.

Classical simulation methods

- thinning Ogata (1981), Cluster/branching representation Hawkes and Oakes (1974)
- **Exact simulation**
- Simulate sequence of jump times of N allowing re-construction of the processes N and λ.

Main drawback both methods have random cost of order $\mathcal{O}(N_T^2)$ due to the non-Markovian Volterra structure and high computational cost for high activity/self-excitation $\Rightarrow$ large N_T $\Rightarrow$ expensive simulations, even for one sample path. Making Monte-Carlo prohibitive!

Can Hawkes processes be simulated ***efficiently***
with ***deterministic*** *complexity?*
In particular, in ***high activity*** *regimes?*

YES, thanks to two key observations

1. **Only integrated quantities matter** Once the integrated intensity $\Lambda_{t_i,t_{i+1}} := \int_{t_i}^{t_{i+1}} \lambda_s ds$ is known on a deterministic interval $[t_i, t_{i+1})$, the increment. $N_{t_{i+1}} - N_{t_i}$ is simply Poisson$(\Lambda_{t_i,t_{i+1}})$.

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2. **Hawkes** $\in$ {**Affine Volterra with jumps**} AJ (2021); Bondi, Livieri, and Pulido (2024); Cuchiero and Teichmann (2019). Focus on the integrated intensity Λ (not λ). Integrating and using Fubini:

$$\Lambda_t = \int_0^t g_0(s)\, ds + \int_0^t K(t-s)(\Lambda_s + Z_s)\, ds,$$

where $Z := N - \Lambda$ is a martingale with $\langle Z \rangle_t = \Lambda_t$. This gives an affine Volterra structure, yielding the characteristic function of Λ via a deterministic nonlinear Volterra equation:

$$\Psi(t) = \int_0^t K(t-s)\, F(\Psi(s))\, ds\,, \quad t \leq T\,.$$

with

$$F(u) = w + e^u - 1\,, \quad u \in \mathbb{C}\,,$$

Fix a discrete time grid $t_0 < t_1 < \ldots < t_n = T$, $t_i = iT/n$.

Characteristic function of Λ

$$\mathbb{E}\left[\exp\left(w\left(\Lambda_{t^n_{i+1}} - \Lambda_{t^n_i}\right)\right)\,\middle|\,\mathcal{F}_{t^n_i}\right] = \exp\left(\int_{t^n_i}^{t^n_{i+1}} F(\Psi(t^n_{i+1} - s))\, dG_{t^n_i}(s)\right), \mathfrak{Re}(w) \leq 0$$

where $G_{t_i}(s)$ is $\mathcal{F}_{t_i}$-measurable and given by $G_{t_i}(s) := \int_{t_i}^{s} g_0(r)\,dr + \int_0^{t_i}\int_{t_i}^{s} K(u-r)\,du\,dN_r$.

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Using the $\Delta t = \frac{T}{n}$, we make the approximations to get a 2nd order equation on $\Psi(\Delta)$ similar as before:

$$\Psi(\Delta t) \approx \int_0^{\Delta t} K(s)\,ds\,F(\Psi(\Delta t)) = \int_0^{T/n} K(s)ds\,F(\Psi(\Delta t)), \quad F(u) = w + e^u - 1 \approx w + u + \frac{u^2}{2},$$

Fix a discrete time grid $t_0 < t_1 < \ldots < t_n = T$, $t_i = iT/n$.

Characteristic function of Λ

$$\mathbb{E}\left[\exp\left(w\left(\Lambda_{t^n_{i+1}} - \Lambda_{t^n_i}\right)\right) \middle| \mathcal{F}_{t^n_i}\right] = \exp\left(\int_{t^n_i}^{t^n_{i+1}} F(\Psi(t^n_{i+1} - s))\, dG_{t^n_i}(s)\right), \ \mathfrak{Re}(w) \leq 0$$

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$$\int_{t^n_i}^{t^n_{i+1}} F(\Psi(t^n_{i+1} - s))\, dG_{t^n_i}(s) \approx F(\Psi(\Delta t))\, G_{t^n_i}(t^n_{i+1}) \approx F(\Psi(\Delta t))\, \alpha^n_i,$$

$$\alpha^n_i = \int_{t^n_i}^{t^n_{i+1}} g_0(s) ds + \sum_{j=0}^{i-1} k^n_{i-j}\, \widehat{N}^n_{j,j+1}, \quad k^n_j = \int_{t^n_j}^{t^n_{j+1}} K(s)\, ds.$$

Fix a discrete time grid $t_0 < t_1 < \ldots < t_n = T$, $t_i = iT/n$.

Characteristic function of Λ

$$\mathbb{E}\left[\exp\left(w\left(\Lambda_{t_{i+1}^n} - \Lambda_{t_i^n}\right)\right)\Big|\mathcal{F}_{t_i^n}\right] = \exp\left(\int_{t_i^n}^{t_{i+1}^n} F(\Psi(t_{i+1}^n - s))\, dG_{t_i^n}(s)\right), \ \mathfrak{Re}(w) \leq 0$$

where $G_{t_i}(s)$ is $\mathcal{F}_{t_i}$-measurable and given by $G_{t_i}(s) := \int_{t_i}^{s} g_0(r)\, dr + \int_0^{t_i}\int_{t_i}^{s} K(u-r)\, du\, dN_r$.

Combining all of our approximations, we obtain

$$\mathbb{E}\left[\exp\left(w\left(\Lambda_{t_{i+1}^n} - \Lambda_{t_i^n}\right)\right)\Big|\mathcal{F}_{t_i^n}\right] \approx \exp\left(\widehat{\Psi}(\Delta t)\frac{\alpha_i^n}{k_0^n}\right),$$

which corresponds to the characteristic functional of an Inverse Gaussian random variable ξ_i^n with parameters $\left(\frac{\alpha_i^n}{1-k_0^n}, \left(\frac{\alpha_i^n}{k_0^n}\right)^2\right)$.

Algorithm - The Hawkes iVi scheme: Simulation of $\widehat{\Lambda}\,,\,\widehat{N}$

1: Compute $k_j^n = \int_{t_j^n}^{t_{j+1}^n} K\,(s)\;ds\,, \quad j = 0\,,\ldots\,,n-1.$

2: **for** $i = 0$ to $n-1$ **do**

3: Compute the quantity: $\alpha_i^n = \int_{t_i^n}^{t_{i+1}^n} g_0(s)ds + \sum_{j=0}^{i-1} k_{i-j}^n\,\widehat{N}_{j,j+1}^n$, with $\alpha_0^n = \int_0^{t_1^n} g_0(s)ds$, for $i = 0$.

4: Simulate the increment:

$$\widehat{N}_{i,i+1}^n \sim \mathcal{P}\left(\xi_i^n\right)\,, \quad \text{where} \quad \xi_i^n \sim IG\left(\frac{\alpha_i^n}{1-k_0^n}\,,\left(\frac{\alpha_i^n}{k_0^n}\right)^2\right).$$

5: Compute the increment:

$$\widehat{\Lambda}_{i,i+1}^n = \alpha_i^n + k_0^n\,\widehat{N}_{i,i+1}^n\,.$$

6: Optional step (jump times): Simulate $\widehat{N}_{i,i+1}^n$ i.i.d. uniform random variables on $[t_i, t_{i+1})$, sort, and append them to $\mathcal{T}^n$.

7: **end for**

Define the piecewise constant càdlàg processes coming from the iVi Hawkes scheme

$$\Lambda^n_t := \sum_{i=0}^{\lfloor nt/T \rfloor - 1} \widehat{\Lambda}^n_{i,i+1}, \quad N^n_t := \sum_{i=0}^{\lfloor nt/T \rfloor - 1} \widehat{N}^n_{i,i+1}, \quad t \le T, \quad n \ge 1.$$

Key Observation (Λ^n, N^n) solve a stochastic Volterra equation with a **measure-valued** kernel:

$$\Lambda^n_t = \int_0^{\lfloor \frac{nt}{T} \rfloor \frac{T}{n}} g_0(s)\, ds + \int_{[0,t]} N^n_{t-s}\, K^n(ds), \quad t \le T.$$

with $K^n(ds) := \sum_{i=0}^{n-1} k^n_i\, \delta_{t^n_i}(ds)$ on $[0, T]$.

Theorem: Convergence

We have the weak convergence

$$(\Lambda^n, N^n) \overset{n\to\infty}{\Longrightarrow} (\Lambda, N),$$

in the Skorokhod J_1-topology, where N is a Hawkes process on $[0, T]$ with exogenous intensity g_0 and memory kernel K, and Λ is its integrated intensity.

For the numerical illustration, we will use the gamma kernel

$$K_\Gamma(t) = c e^{-bt} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad b > 0,\ \alpha > 0,\ c > 0,\ t \geq 0,$$

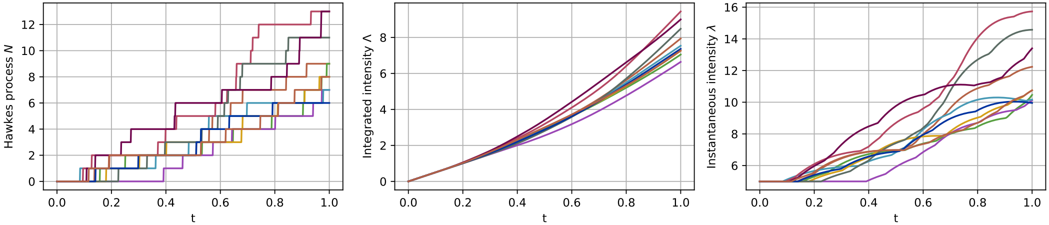
with the parameters $b = 3,\ \alpha = 2,\ c = 0.9 b^\alpha$ and $\mu = 5,\ T = 1$.

- when α is an integer, the gamma kernel reduces to the Erlang kernel widely used in the applications of the Hawkes processes.

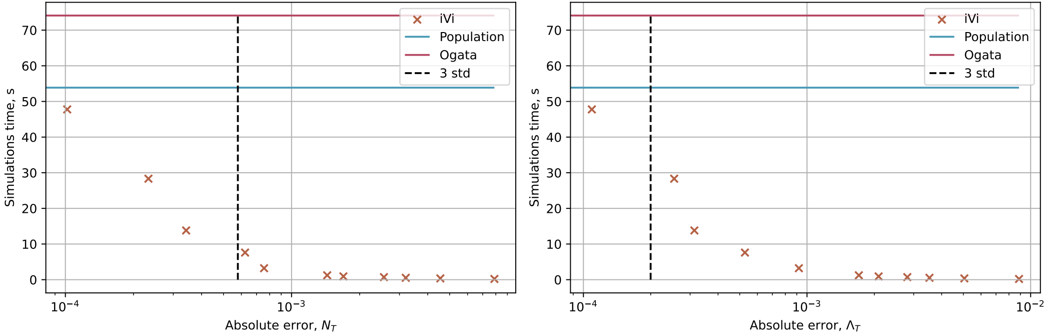
Sample trajectories of N, Λ and the instantaneous intensity λ generated with the iVi scheme with time step 0.001. The instantaneous intensity is computed using the left-endpoint approximation

$$\lambda_{t_i} = g_0(t_i) + \int_0^{t_i} K(t_i - s)\, dN_s = g_0(t_i) + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} K(t_i - s)\, dN_s \approx g_0(t_i) + \sum_{j=0}^{i-1} K(t_i - t_j) \widehat{N}^n_{j,j+1},$$

for $i = 0, \ldots, n$.

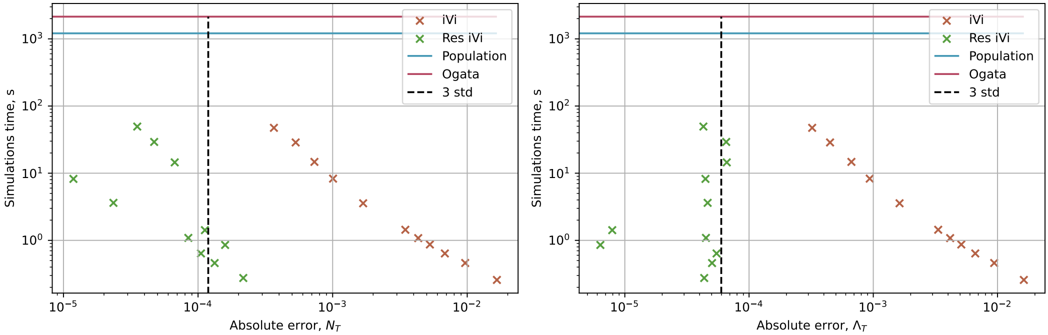


Sample trajectories simulated with the iVi scheme with time step 0.001.



Simulation time for 10^5 trajectories (y-axis) versus the absolute error of the Monte Carlo estimator of the Laplace transform (x-axis). Crosses represent estimators obtained with the iVi scheme under different discretization steps. Horizontal bars indicate the exact methods: Population (blue) and Ogata (purple).

Can be boosted using a **Resolvent version of iVi**



Wrap-up

We had two key questions

- **Mathematical** Can we reconcile the affine structure with the Volterra affine processes dynamics for simulation?
- **Practical** Can we design a scheme that performs well in challenging market regimes with very few discretization steps?

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We had two key questions

- **Mathematical** Can we reconcile the affine structure with the Volterra affine processes dynamics for simulation?
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iVi schemes answer positively both questions

- Reconciling affine structure:
  - In integrated form, the quadratic variation of Z is affine and equal to U, no more square-root for Heston, deterministic complexity for Hawkes!
  - A right end point rule on the dynamics of U and on the Riccati equations for the characteristic function lead to the same approximation!
- Challenging market and high activity regimes are usually very close to very jumpy behaviour (limiting IG in Heston). iVi scheme efficiently captures this by design in Heston and Hawkes.

Motto

If it looks like a jump, simulate it like a jump!

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(Alfonsi, 2025, Definition 2.1)

Let $K : \mathbb{R}_+ \to \mathbb{R}_+$ be a kernel such that $K(0) > 0$. We say that the kernel K preserves nonnegativity if, for any $i \geq 1$, any $x_1, \dots, x_i \in \mathbb{R}$, and $0 \leq t_1 < \cdots < t_i$ such that:

$$\sum_{j=1}^{\ell} K(t_\ell - t_j)x_j \geq 0, \quad \ell = 1, \dots, i,$$

we have:

$$\sum_{j : t_j \leq t} K(t - t_j)x_j \geq 0, \quad t \geq 0,$$

with the convention $\sum_{\emptyset} = 0$.

Integrated quantities

$$\bar{g}_0(t) := \int_0^{\frac{T}{n}} g_0(t+s)\,ds \quad \text{and} \quad \bar{K}(t) := \int_0^{\frac{T}{n}} K(t+s)\,ds,$$

Theorem

Let $K, g_0 \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$, $b \leq 0$ and $c \geq 0$. Assume that $\bar{g}_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and nonnegative and that $\bar{K} : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing and preserves nonnegativity. Consider $(\widehat{U}_{i,i+1})_{i=0,\ldots,n-1}$, and $(\widehat{Z}_{i,i+1})_{i=0,\ldots,n-1}$ satisfying the recursions of the iVi *scheme. Then, we have that*

$$\alpha_i, \widehat{U}_{i,i+1} \geq 0, \quad i = 0, \ldots, n-1,$$

and the algorithm is well-defined.

Construct the piecewise constant processes

$$U_t^n := \sum_{i=0}^{\lfloor nt/T \rfloor - 1} \widehat{U}^n_{i,i+1}\,, \quad Z_t^n := \sum_{i=0}^{\lfloor nt/T \rfloor - 1} \widehat{Z}^n_{i,i+1}\,, \quad t \le T\,, \quad n \ge 1\,.$$

Key observation (U^n, Z^n) solve a stochastic Volterra equation with a **measure-valued** kernel:

$$U_t^n = \int_0^{\lfloor \frac{nt}{T} \rfloor \frac{T}{n}} g_0(s)\,ds + \int_{[0,t]} \left(b\, U^n_{t-s} + c\, Z^n_{t-s}\right) K^n(ds) \quad t \le T\,.$$

with $K^n(ds) := \sum_{i=0}^{n-1} k_i^n\, \delta_{t_i^n}(ds)$ on $[0\,,\,T]$.

Assumption For any $n \geq 1$, $\bar{K}^n$ is non-increasing and preserves nonnegativity, and $\bar{g}_0^n$ is non-decreasing and nonnegative.

Theorem

Let $K, g_0 \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}_+)$. Then, the sequence $(U^n, Z^n)_{n\geq 1}$ is $J_1(\mathbb{R}^2)$-tight on $D([0,T])^2$. Moreover, any accumulation point (U, Z) satisfies the following:

1. *U is a continuous, non-decreasing process, starting from 0.*
2. *Z is a continuous square-integrable martingale with respect to the filtration generated by (U, Z), starting from 0, such that $\langle Z \rangle_t = U_t, \quad t \leq T$.*
3. *The following Volterra equation holds:*

$$U_t = \int_0^t g_0(s)\, ds + \int_0^t K(t-s)\,(b\, U_s + c\, Z_s)\, ds, \quad t \leq T, \quad a.s.$$