Multivariate Self-Exciting Processes with Dependencies (MSPD)- Application to Cyber Risk

Caroline Hillairet, ENSAE IP Paris, CREST

Joint works with Y. Cherkaoui (CREST and Milliman)
T. Peyrat (CREST and Exiom)
A. Réveillac (Institut de Mathématiques de Toulouse)

Advances in Financial Mathematics 2026

Paris, 30 January 2026

Outline

Cumulative Loss process

$$L_t := \sum_{i=1}^{N_t} Y_i, \quad t \in [0, T],$$

- Frequency: Claims arrival modeled by a counting process $N := (N_t)_{t\in[0,T]}$, jumping at time $(\tau_i)_{i\in\mathbb{N}^*}$,
- Severity: claims sizes $(Y_i)_{i\in\mathbb{N}^*}$

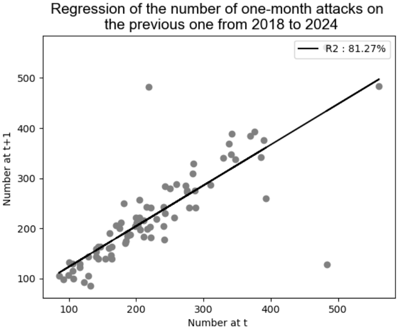
Classical **Cramer-Lundberg** model

- N is a Poisson process (inter-arrivals $(\tau_i - \tau_{i-1})$ are iid)
- N is independent of the claims sizes (Y_i),
- (Y_i) iid random variables.

CL $\Rightarrow$ Tractable computations due to the independence assumptions $\cdots$ but they are not satisfied for cyber risk (and systemic risk).

Autocorrelation of the number of cyber-events

- **Hackmageddon database**, maintained by Paulo Passeri (more than 19 000 events over 2018-2024)
- Regression of the number of one-month $(t+1)$ events on the previous month t (should be independent for a Poisson process model to be valid)



- Autocorrelation in other cyber-database (**PRC database**, see [BBH20])

Relaxing independence assumptions on claims arrivals

Self-exciting arrival of claims and clustering effects

- Hawkes process: self-exciting counting process with stochastic intensity, fully specified by the point process itself (equivalently its jump times $(\tau_n)_n$)
- Many papers dedicated to Hawkes processes (among many others)
  - in finance: LOB as in Bacry et al. (2013) or Rosenbaum et al. (2015), Credit risk as in Errais et al. (2013) or Bielecki et al. (2020)...
  - in insurance: Dassios and Zhao (2012), Magnusson (2015), Gao and Zhu (2018), Swishchuk (2018), Brachetta et al. (2022)..
- H (linear) Hawkes process with
  - (deterministic) excitation kernel Φ
  - (deterministic) baseline intensity μ

  is the counting process ($H_0 = 0$) with intensity process

$$\lambda_t := \mu(t) + \int_{(0,t)} \Phi(t-s)dH_s = \mu(t) + \sum_{\tau_n < t} \Phi(t-\tau_n)$$

Impact of the severity of the claims on the excitation kernel

Cumulative Loss process

$$L_t := \sum_{i=1}^{H_t} Y_i, \quad t \in [0, T],$$

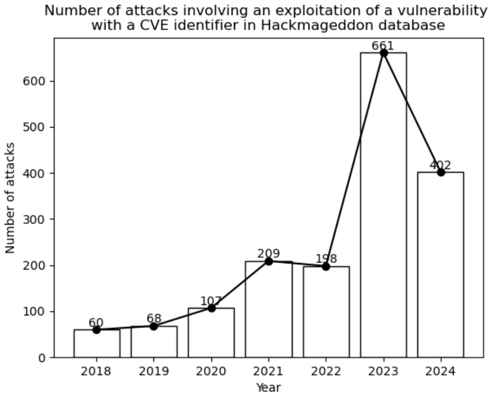
- Frequency: Claims arrival modeled by a Hawkes process H
- Impact of claims amounts on the excitation kernel : dependencies between the frequency component H and the severity component (claims amounts Y_i)

$$\lambda_t := \mu(t) + \sum_{\tau_n < t} b(Y_{\tau_n}) \, \Phi(t - \tau_n)$$

for a (deterministic) function b s.t. $\mathbb{E}(b(Y)) \|\Phi\|_1 < 1$.

External excitation

- The clustering feature not only due to contagion, but also to the arrival of external shocks such as **vulnerabilities** disclosures
- In Hackmageddon database : Number of cyber-attacks involving a vulnerability with a CVE identifier



Stochastic baseline intensity

- Extension of Hawkes with stochastic baseline intensity μ
- The stochastic baseline intensity captures the impact of random external events, such as **vulnerabilities** disclosure

$$\lambda_t = \underbrace{\mu_0(t)}_{deterministic\ part} + \underbrace{\sum_{\tau_k^{ext}<t} \phi^{ext}(t-\tau_k^{ext}, Y_k^{ext})}_{External\ shocks} + \underbrace{\sum_{\tau_n^{int}<t} \Phi^{int}(t-\tau_n^{int}, Y_n^{int})}_{Self-excitation} \tag{1}$$

- External shocks (vulnerabilities) counting process $N_t^{ext} = \sum_{k\geq 1} 1_{\tau_k^{ext}\leq t}$ (Poisson process or self-exciting process)
- with severity marks $\{Y_k^{ext}\}_{k\geq 1}$ i.i.d. positive random variables
  - For cyber vulnerabilities: CVSS scores notified in the National Vulnerability Database

Multivariate self-exciting processes with dependencies

- A **more accurate**/flexible model (than Cramer-Lundberg) ... but **less tractable**
- How to compute formula for Risk Evaluation (such as expectation, correlation, expected surplus loss of stress scenarios, expected shortfall, insurance Stop Loss contracts...)?
  - For general kernels Φ (with spectral radius < 1)
  - In no stationary regimes
- Towards an unifying writing : Poisson imbedding also called Thinning Algorithm - Ogata (1981), Brémaud-Massoulié (1996) ...
  - A general class of processes: Multivariate self-exciting processes with dependencies (MSPD)
  - Multivariate version allows one to take into account risks with multivariate components, external excitation.
- Applications: Cyber risk, Credit risk, Microstructure (LOB).
- Tools :
  - Malliavin IPP formula (Last (2016))
  - Pseudo-chaotic expansion (H., Réveillac (2024))

Outline

1 Loss process with dependencies

2 Multivariate self-exciting processes with dependencies

3 Risk quantification and stress scenarios

Thinning Algorithm (for a standard Hawkes)

A Hawkes process with intensity $\lambda_t = \mu + \sum_{\tau_n < t} \Phi(t - \tau_n)$ can be represented in terms of a Poisson measure N on $[0, T] \times \mathbb{R}_+$ with compensator $dt \otimes d\theta$

$$\begin{cases} H_t = \int_{(0,t]} \int_{\mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_s\}} N(ds, d\theta), \\ \lambda_t = \mu + \int_{(0,t)} \Phi(t-s) dH_s = \mu + \int_{(0,t)} \int_{\mathbb{R}_+} \Phi(t-s) \mathbf{1}_{\{\theta \leq \lambda_s\}} N(ds, d\theta). \end{cases}$$

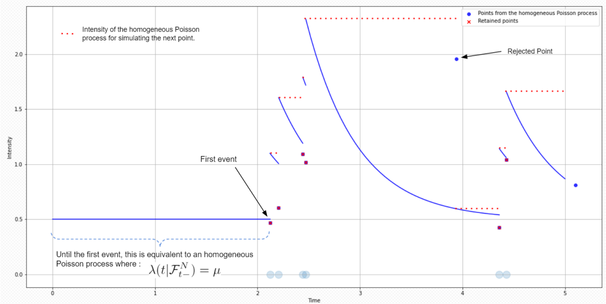


Illustration from T. Peyrat ($\mu = 0.5$, exp kernel)

(Univariate) self-exciting processes with dependencies

Representation through the imbedding procedure in terms of a Poisson measure $N(ds, d\theta, dy)$ on $\mathbb{X} := [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$; $\quad x := (t, \theta, y) \in \mathbb{X}$ with compensator $\rho(dx) = dt \otimes d\theta \otimes \nu(dy)$

$$\begin{cases} Z_T^{\mu,\zeta,\Phi} := \int_{[0,T]\times\mathbb{R}_+\times\mathbb{R}_+} \zeta(T-t,y)\mathbf{1}_{\{\theta\leq\lambda_t\}} N(dt,d\theta,dy) \\ \text{with the intensity given by} \\ \lambda_T = \mu(T) + \int_{[0,T]\times\mathbb{R}_+\times\mathbb{R}_+} \Phi(T-t,y)\mathbf{1}_{\{\theta\leq\lambda_t\}} N(dt,d\theta,dy). \end{cases} \tag{2}$$

so that

$$Z_T^{\mu,\zeta,\Phi} = \begin{cases} L_T, & \text{if } \zeta(u,y) = y \ \texttt{(Loss process)} \\ H_T, & \text{if } \zeta(u,y) = 1 \ \texttt{(Counting process)} \\ \lambda_T - \mu(T), & \text{if } \zeta(u,y) = \Phi(u,y) \\ & \texttt{(Excitation component of the intensity process).} \end{cases}$$

Multivariate self-exciting processes with dependencies

Mutivariate version defined through the imbedding procedure with respect to a d-dimensional Poisson measure $\mathrm{N}(dk, dt, d\theta, dy)$: for each component $\boldsymbol{Z}^i_T$, $i = 1, \cdots, d$

$$\begin{cases} \boldsymbol{Z}^i_T = \int_{\{1,\dots,d\}\times(0,T]\times\mathbb{R}^2_+} \zeta^{i,k}(T-t,y)\mathbf{1}_{\{\theta\le\lambda^k_t\}}\mathrm{N}(dk,dt,d\theta,dy) \\ \boldsymbol{\lambda}^i_T := \boldsymbol{\mu}^i(t) + \int_{\{1,\dots,d\}\times(0,T]\times\mathbb{R}^2_+} \Phi^{i,k}(T-t,y)\mathbf{1}_{\{\theta\le\lambda^k_t\}}\mathrm{N}(dk,dt,d\theta,dy) \end{cases}$$

- $\Phi^{i,k}$: impact of the k^{th}-dimension on the i^{th}-dimension
- Multivariate version allows one to take into account risks with multivariate components, external excitation...
- The Loss process for model (1) (external excitation arriving as a Poisson process with intensity ρ) corresponds to kernels

$$\boldsymbol{\zeta} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} \mu_0 \\ \rho \end{pmatrix}, \ \Phi(u,y) := \begin{pmatrix} \Phi^{int}(u,y) & \Phi^{ext}(u,y) \\ 0 & 0 \end{pmatrix}.$$

Outline

Malliavin IPP formula (Mecke formula)

For any random variable F in $L^2(\Omega)$ (for example $F = \mathbb{I}_{L_T \geq K}$)

$$\mathbb{E}\left[F \int_{\mathbb{X}} h(x)\,\mathrm{N}(dx)\right] = \int_{\mathbb{X}} h(x)\,\mathbb{E}\left[F \circ \varepsilon_x^+\right] \rho(dx) \qquad (3)$$

- $F \circ \varepsilon_x^+$ denotes the functional on the Poisson space where a deterministic point $x = (t, \theta, y)$ is added to the paths of N.
- **adding a jump at some time** $x = (t, \theta, y)$
  $\rightarrow$ **adding "artificially" a claim of size** y **at time** t
  $\rightarrow$ stress scenario
- In the Cramer-Lundberg model: the additional jump only impacts the cumulative loss by adding a new event.
- **In case of a self exciting (Hawkes) process : it also impacts the dynamic (after time** t**) of the counting process.**

Chaotic expansion - Privault (2009), Last (2016)...

Representation of random variable F in $L^2(\Omega)$ as an infinite sum of factorial Poisson integrals.

- $\mathbb{X} := [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+; \quad x := (t, \theta, y); \quad \rho(dx) = dt\, d\theta\, \nu(dy)$
- Malliavin derivative:

$$D_x F := F \circ \epsilon_x^+ - F$$
$$D^n_{(x_1,\ldots,x_n)} F := \sum_{J \subset \{x_1,\ldots,x_n\}} (-1)^{n-|J|} F \circ \varepsilon_J^{+,|J|},$$

- **Standard chaotic expansion** (w.r.t. compensated Poisson measure)

$$F = \mathbb{E}[F] + \sum_{k=1}^{+\infty} \int_{\mathbb{X}^k} \frac{1}{k!} \mathbb{E}\left[D^k_{(x_1,\cdots,x_k)} F\right] (N(dx_1) - \rho(dx_1)) \cdots (N(dx_k) - \rho(dx_k))$$

$\rightarrow \mathbb{E}\left[D^k_{(x_1,\cdots,x_k)} F\right]$ difficult to compute explicitly.
Already for the first coefficient, one need to know the CDF of λ_t.

"Pseudo-Chaotic" expansion [HR24]

- **Explicit "Pseudo-Chaotic" expansion** (w.r.t. Poisson measure)

$$F = \sum_{k=1}^{+\infty} \int_{\mathbb{X}^k} \frac{1}{k!} \underbrace{\mathcal{T}^k_{(x_1,\dots,x_k)} F}_{c_k(x_1,\cdots,x_k)} N(dx_1)\cdots N(dx_k),$$

- Deterministic operators $\mathcal{T}^n$

$$\mathcal{T}^0 F := F(\emptyset),$$

$$\mathcal{T}^n_{(x_1,\cdots,x_n)} F := \sum_{J \subset \{x_1,\cdots,x_n\}} (-1)^{n-|J|} F\left(\sum_{y_i \in J} \delta_{x_i}\right).$$

$\rightarrow$ explicit coefficients $c_k(x_1,\cdots,x_k)$ (no expectation to compute):

$$\text{For ex. if } F = L_T : \begin{cases} c_1(x_1) = y_1 \mathbf{1}_{\{\theta_1 \le \mu\}}, \\ c_2(x_1,x_2) = y_2 \mathbf{1}_{\{\theta_1 \le \mu\}} \mathbf{1}_{\{\mu \le \theta_2 \le \mu + \Phi(t_2 - t_1)\}} \\ \dots \end{cases}$$

Methodology for the computation of MSPDs' moments

- The idea of the computation relies on mixing the pseudo-chaotic expansion and Mecke's formula.
- For $\Gamma \in L^1$

$$\mathbb{E}\left[\Gamma\, Z_T^{\mu,\zeta,\Phi}\right]$$

$$= \sum_{n=1}^{+\infty} \mathbb{E}\left[\Gamma \int_{\mathbb{X}^n} \frac{1}{n!} \mathcal{T}^n_{(x_1,\dots,x_n)} Z_T^{\mu,\zeta,\Phi}\, N(dx_1)\cdots N(dx_n)\right]$$

$$= \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\mathbb{X}^n} \mathcal{T}^n_{(x_1,\dots,x_n)} Z_T^{\mu,\zeta,\Phi} \mathbb{E}\left[\Gamma \circ \varepsilon^{+,n}_{(x_1,\cdots,x_n)}\right] \rho(dx_1)\cdots\rho(dx_n).$$

- Compute each terms, integrating in time makes the iterated convolutions appear, and then sum in n.

First order moment of a (M)SPD

- **For the process $Z_T^{\mu,\zeta,\Phi}$**

$$\mathbb{E}\left[Z_T^{\mu,\zeta,\Phi}\right] = \int_{(0,T)} \bar{\zeta}(T-s)\left(\mu(s) + \int_{(0,s)} \bar{\Psi}(s-u)\mu(u)du\right) ds.$$

  - where $\bar{\zeta}(u) := \int \zeta(u,y)\nu(dy)$
  - $\bar{\Psi} := \sum_{n=1}^{+\infty} \bar{\Phi}_n$ is the resolvent with $\bar{\Phi}_n$ are the iterated convolutions

$$\bar{\Phi}_1(t) := \bar{\Phi}(t) = \int \Phi(u,y)\nu(dy), \quad \bar{\Phi}_n(t) := \int_0^t \bar{\Phi}(t-s)\bar{\Phi}_{n-1}(s)ds.$$

First order moment of the shifted process

- **Stressed trajectories** by adding p shocks $(x_1, \ldots, x_p)$

$$\mathbb{E}[Z_T^{\mu,\zeta,\Phi} \circ \varepsilon^{+,p}_{(x_1,\ldots,x_p)}] = \mathbb{E}[Z_T^{\mu,\zeta,\Phi}] + \sum_{j=1}^{p} \zeta(T - t_j, y_j)$$
$$+ \sum_{j=1}^{p} \int_{t_j}^{T} \bar{\zeta}(T-s) \left(\Phi(s - t_j, y_j) + \int_{t_j}^{s} \bar{\Psi}(s-u)\Phi(u - t_j, y_j) du \right) ds.$$

- The last line:
  - Impact of aftershocks (=0 in a Cramer Lundberg model)
  - We recognize the expectation of a MSPD with the baseline μ replaced by the excitation kernel Φ taken at the deterministic jump.

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Application to cyber portfolio: stress scenarios

- Model (1)

$$\lambda_t = \underbrace{\mu_0(t)}_{deterministic\ part} + \underbrace{\sum_{\tau_k^{ext}<t} \phi^{ext}(t-\tau_k^{ext}, Y_k^{ext})}_{External\ shocks} + \underbrace{\sum_{\tau_n^{int}<t} \Phi^{int}(t-\tau_n^{int}, Y_n^{int})}_{Self-excitation}$$

- Stress scenario $\mathcal{S}^p$ = set of added points $\{x_1, \dots, x_p\}$ such that for every $x_i := (k_i, t_i, \theta_i, y_i) \in S^p$, $\theta_i = 0$.
- Two stress scenarios
  - an excess of claims in the portfolio : $k_i = 1$
  - a massive disclosure of critical vulnerabilities: $k_i = 2$
- **Closed-form formulas for the expected surplus of Loss**
  - For the Poisson case

$$\mathbb{E}[L_T \circ \varepsilon^{+p}_{\mathcal{S}^p}] - \mathbb{E}[L_T] = \sum_{j=1}^{p} y_j \mathbf{1}_{\{k_j=1\}}$$

  - For Hawkes model (1): additional terms due to the aftershocks

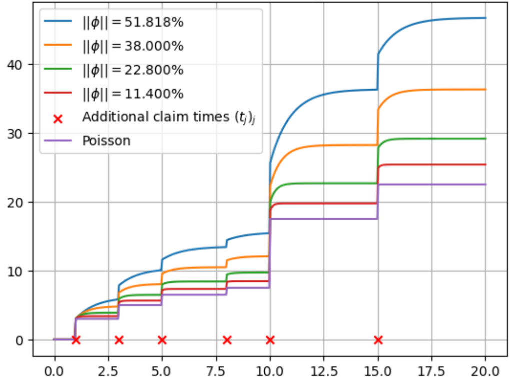
Stress scenarios with additional claims

Exponential kernels, with parameters calibrated on real data [BCH25].

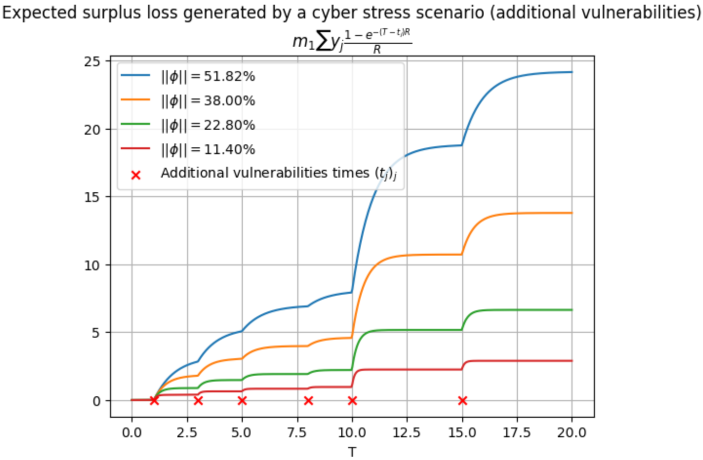
λ_0	ρ	$m_2 = \mathbb{E}(Y_2)$	$m_1 = \mathbb{E}(Y_1)$	δ	$R = \delta - m_1$
3.10	84.79	0.12	0.83	2.37	1.54

Expected surplus loss generated by a cyber stress scenario (additional claims)

$$\sum y_j + m_1 \sum y_j \frac{1 - e^{-(T-t_j)R}}{R}$$



Stress scenarios with additional vulnerabilities



Second order moment

- **Explicit covariance for $Z^{\mu,\zeta,\Phi}$, $Z^{\tilde{\mu},\tilde{\zeta},\tilde{\Phi}}$:**

$$Cov\left(Z_T^{\mu,\zeta,\Phi}, Z_S^{\tilde{\mu},\tilde{\zeta},\tilde{\Phi}}\right) = \mathbb{E}\left[Z_T^{\mu,\zeta\tilde{\rho},\Phi}\right] + \int_0^T \bar{\zeta}(T-v)\left(\mathbb{E}[Z_v^{\mu,\Phi\tilde{\rho},\Phi}] + \int_0^v \bar{\Psi}(v-w)\mathbb{E}[Z_w^{\mu,\Phi\tilde{\rho},\Phi}]dw\right) dv$$

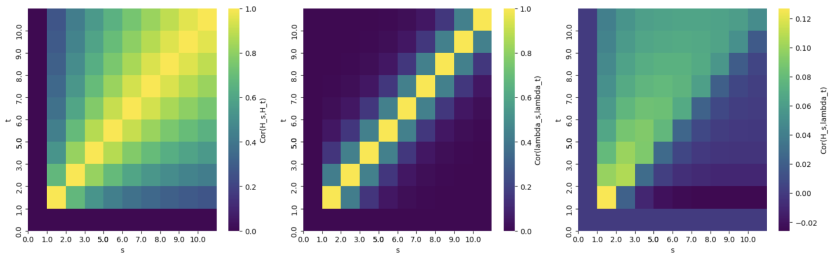
- Ex: Hawkes with dependencies, for $S \leq T$, denoting $C := \frac{\mathbb{E}[b(Y)^2]}{\mathbb{E}[b(Y)]^2}$:

$$Cov\left(H_T, H_S\right) = \int_0^T \left(1 + \int_u^S \bar{\Psi}(v-u)dv\right)\left(\int_0^u \bar{\Psi}(u-v)\mu(v)dv + \mu(u)\right) du$$
$$+ \int_0^T\int_u^T \bar{\Psi}(y-u)dy\left(1 + C\int_u^S \bar{\Psi}(v-u)dv\right)\left(\int_0^u \bar{\Psi}(u-v)\mu(v)dv + \mu(u)\right)du$$

- Similar expressions for $\text{Cov}(\lambda_s, \lambda_t)$, $\text{Cov}(L_s, L_t)$, $\text{Cov}(H_s, \lambda_t)$, $\text{Cov}(L_s, \lambda_t)$

Correlations - Heat Map

- Correlations of Hawkes and its intensity (exponential kernel)



- the correlation is more persistant for H than for λ
- the correlation of λ on H decreases more rapidly than the reciprocal correlation

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Conclusion

- General model tacking into account self-excitation, external excitation, and impact of the severity component (size of the claims) in the frequency component (contagion).
- Efficient methodology, combining Malliavin calculus and Poisson imbedding, to obtain closed-form formula for the computation of general loss processes functionals (expectation, variance and correlation), for general kernels.
- Works in progress :
  - Computations of Stop Loss contracts and Expected Shortfall.
  - Heterogeneity in cyber-risk: Classification of cyber events through CART trees, for clustering Hawkes process trajectories.

Thank you for your attention.

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