Deep Signature Approach for McKean-Vlasov FBSDEs in a Random Environment

Mean-field games with common noise provide a powerful framework for modeling the collective behavior of large populations subject to shared randomness, such as systemic risk in finance or environmental shocks in economics. These problems can be reformulated as McKean-Vlasov forward-backward stochastic differential equations (MV-FBSDEs) in a random environment, where the coefficients depend on the conditional law of the state given the common noise. Existing numerical methods, however, are largely limited to cases where interactions depend only on expectations or low-order moments, and therefore cannot address the general setting of full distributional dependence. In this work, we introduce a deep learning-based algorithm for solving MV-FBSDEs with common noise and general mean-field interactions. Building on fictitious play, our method iteratively solves conditional FBSDEs with fixed distributions, where the conditional law is efficiently represented using signatures, and then updates the distribution through supervised learning. Deep neural networks are employed both to solve the conditional FBSDEs and to approximate the distribution-dependent coefficients, enabling scalability to high-dimensional problems. Under suitable assumptions, we establish convergence in terms of the fictitious play iterations, with error controlled by the supervised learning step. Numerical experiments, including a distribution-dependent mean-field game with common noise, demonstrate the effectiveness of the proposed approach.